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On Dvoretzky's theorem for subspaces of L_p

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We prove that for any $2 < p < \infty$ and for every n-dimensional subspace X of L_p, represented on $\mathbb{R}^n$, whose unit ball B_X is in Lewis' position one has the following two-level Gaussian concentration inequality:

$$\mathbb{P}\left(\big|\|Z\| - \mathbb{E}\|Z\|\big| > \varepsilon\mathbb{E}\|Z\|\right) \leq C\exp\left(-c\min\left\{\alpha_p\varepsilon^2 n, (\varepsilon n)^{2/p}\right\}\right), \quad 0 < \varepsilon < 1,$$

where Z is the standard n-dimensional Gaussian vector, $\alpha_p > 0$ is a constant depending only on p and $C, c > 0$ are absolute constants. As a consequence we show optimal lower bound on the dimension of random almost spherical sections for these spaces. In particular, for any $2 < p < \infty$ and every n-dimensional subspace X of L_p, the Euclidean space ℓ_2^k can be $(1+\varepsilon)$-embedded into X with $k \geq c_p \min\{\varepsilon^2 n, (\varepsilon n)^{2/p}\}$, where $c_p > 0$ is a constant depending only on p. This improves upon the previously known estimate due to Figiel, Lindenstrauss and Milman.

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1. Introduction

In the present note we study the classical result of Dvoretzky [9] on almost spherical sections of normed spaces in the case of subspaces of L_p. Grothendieck in [17], motivated by the well known Dvoretzky–Rogers lemma from [10], asked if every finite-dimensional normed space has lower dimensional subspaces which are almost Euclidean and their dimension grows with respect to the dimension of the ambient space. Dvoretzky in [9] gave an affirmative answer in the above question by proving that for any positive integer k and every $\varepsilon \in (0,1)$ there exists $N = N(k,\varepsilon)$ with the following property: For every $n \geq N$ and any n-dimensional normed space X there exists a k-dimensional subspace E which is $(1+\varepsilon)$-isomorphic to the Euclidean space ℓ_2^k. In modern functional analytic language this means that every infinite-dimensional Banach space contains ℓ_2^n's uniformly. Dvoretzky's proof in [9, Theorem 1] provides the quantitative estimate $N(k,\varepsilon) \geq \exp(c\varepsilon^{-2}k^2\log^2 k)$ (see [39] for a related discussion), for some absolute constant $c > 0$.[3] However, the aforementioned estimate is not optimal. The optimal dependence with respect to the dimension was proved later by Milman, in his groundbreaking work [27], where he obtained $N(k,\varepsilon) \geq \exp(ck\varepsilon^{-2}\log\frac{1}{\varepsilon})$ (an alternative approach which yields the same estimate was presented by Szankowski in [39]). Equivalently, this states that for any $\varepsilon \in (0,1)$ there exists a function $c(\varepsilon) > 0$ with the following property: for every n-dimensional normed space X there exists $k \geq c(\varepsilon)\log n$ and a linear map $T : \ell_2^k \to X$ with $\|x\|_2 \leq \|Tx\|_X \leq (1+\varepsilon)\|x\|_2$ for all $x \in \ell_2^k$. In this case we say that ℓ_2^k can be $(1+\varepsilon)$-embedded into X or that X has a k-dimensional subspace which is $(1+\varepsilon)$-Euclidean and we write $\ell_2^k \overset{1+\varepsilon}{\hookrightarrow} X$.

The example of $X = \ell_\infty^n$ shows that this result is best possible with respect to n (see [27] or [12, Proposition 3.2] for the details). The approach of [27] is probabilistic in nature and provides that the vast majority of subspaces (in terms of the Haar probability measure on the Grassmannian manifold $G_{n,k}$) are $(1+\varepsilon)$-spherical, as long as $k \leq c(\varepsilon)k(X)$, where $k(X)$ is the *critical dimension* of X (see below for the definition). Nowadays this is customary addressed as the randomized Dvoretzky theorem or random version of Dvoretzky's theorem. Milman in this work revealed the significance of the concentration of measure as a basic tool for the understanding of the high-dimensional structures. That was the starting point for many applications of the concentration of measure method in high-dimensional phenomena. Since then, this tool has found numerous applications in various fields such as quantum information [4], combinatorics [7], random matrices [42], compressed sensing [13], theoretical computer science [26], geometry of high-dimensional probability measures [11] and more.

Another remarkable fact of Milman's approach is that the critical quantity $k(X)$ can be described in terms of the global parameters of the space. In particular, $k(X) \simeq \mathbb{E}\|Z\|_X^2/b^2(X)$ where Z is a standard Gaussian random vector in X and

[3] Here and elsewhere in this paper c and C denote positive absolute constants, not necessarily the same at each occurrence.