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A family of functional inequalities: Łojasiewicz inequalities and displacement convex functions

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ABSTRACT

For displacement convex functionals in the probability space equipped with the Monge–Kantorovich metric we prove the equivalence between the gradient and functional type Łojasiewicz inequalities. We also discuss the more general case of λ-convex functions and we provide a general convergence theorem for the corresponding gradient dynamics. Specialising our results to the Boltzmann entropy, we recover Otto–Villani's theorem asserting the equivalence between logarithmic Sobolev and Talagrand's inequalities. The choice of power-type entropies shows a new equivalence between Gagliardo–Nirenberg inequality and a nonlinear Talagrand inequality. Some nonconvex results and other types of equivalences are discussed.

1. Introduction

Łojasiewicz inequalities are known to be extremely powerful tools for studying the long-time behaviour of dissipative systems in an Euclidean or Hilbert space, see e.g., [33, 22,15,8] and references therein. Their connection with the asymptotics of gradient flows

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comes from the fact that one of this inequality asserts that the underlying energy can be rescaled near critical points into a sharp function.[1] A consequence of this inequality is that gradient curves can be shown to have finite length through the choice of an adequate Lyapunov function, see [28,25,8].[2]

In parallel the study of large time asymptotics of various PDEs, also based on energy techniques, was developed in close conjunction with functional inequalities. A classical study protocol is to evidence Lyapunov functionals and use functional inequalities to derive quantitative contractive properties of the flow. The heat equation provides an elementary but illustrative example of this approach: the Boltzmann entropy gives a Lyapunov functional while the Logarithmic-Sobolev inequality ensures the exponential convergence of the solution curve to a self-similar profile. Numerous applications of these techniques, as well as their stochastic counterparts, can be found in e.g., [4,13,3,16,6,18]. Standard references for functional inequalities are for instance [12,27,26,21].

In this article we show that the joint use of metric gradient flows and Łojasiewicz inequalities allows for a systematic and transparent treatment of these evolution equations. In this regard the "Riemannian structure" of the set of probability measures endowed with the Monge–Kantorovich distance (see [30,2,19,35,24]) plays a fundamental role in our approach. It allows in particular to interpret some PDEs as gradient flows, like Fokker–Planck, porous medium, or fast diffusion equations, and it provides a setting sufficiently rich to formulate precisely Łojasiewicz inequalities.

In a first step we indeed introduce two types of Łojasiewicz inequalities in the probability space equipped with the Monge–Kantorovich distance. One of these inequalities is a growth measure of the energy functional with respect to the Monge–Kantorovich distance to stationary points, while the other provides a relationship between the values of the energy and its slope. The latter is called the *gradient Łojasiewicz inequality*. In this functional setting, both inequalities can be viewed as families of abstract functional inequalities. We prove their equivalence in the case of convex functionals. Specialising our results to Boltzmann's entropy we recover Otto–Villani's theorem [31] stating the equivalence between the logarithmic Sobolev and Talagrand inequalities. We also prove a new equivalence between Gagliardo–Sobolev inequality and a nonlinear Talagrand type inequality (22). For general λ-convex functionals the gradient Łojasiewicz gradient inequality merely implies the growth around the minimisers set, the reverse implication is in general false. This difficulty can be felt through the fact that Talagrand's inequality is not known to imply logarithmic Sobolev inequality for a general non-convex[3] confinement potential, see [35, Section 9.3.1, p. 292]. In Section 3 we discuss further this issue and provide a new proof that the reverse implication holds for convex potentials with L^∞ variations, [31, Corollary 2.2].

[1] See Remark 1 for further explanations.
[2] See also the proof of Theorem 1.
[3] However it holds for potential whose Hessian is bounded from below, see [31, Corollary 3.1].