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Finite range decomposition for Gaussian measures with improved regularity

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ABSTRACT

We consider a family of gradient Gaussian vector fields on the torus $(\mathbb{Z}/L^N\mathbb{Z})^d$. Adams, Kotecký, Müller and independently Bauerschmidt established the existence of a uniform finite range decomposition of the corresponding covariance operators, i.e., the covariance can be written as a sum of covariance operators supported on increasing cubes with diameter L^k. We improve this result and show that the decay behaviour of the kernels in Fourier space can be controlled. Then we show the regularity of the integration map that convolves functionals with the partial measures of the finite range decomposition. In particular the new finite range decomposition avoids the loss of regularity which arises in the renormalisation group approach to anisotropic problems in statistical mechanics.

1. Introduction

In this paper we consider finite range decompositions for families of translation invariant Gaussian fields on a torus $T_N = (\mathbb{Z}/L^N\mathbb{Z})^d$. A Gaussian process ξ indexed by T_N

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has range M if $\mathbb{E}(\xi(x)\xi(y)) = 0$ for any x, y such that $|x-y| \geq M$. A finite range decomposition of ξ is a decomposition $\xi = \sum_k \xi_k$ such that the ξ_k are independent processes with range $\sim L^k$. Equivalently, if $\mathcal{C}(x,y)$ is the covariance of ξ then a finite range decomposition is possible if there are covariances $\mathcal{C}_k(x,y)$ such that $\mathcal{C} = \sum_k \mathcal{C}_k$, $\mathcal{C}_k(x,y) = 0$ for $|x-y| \gtrsim L^k$, and $\mathcal{C}_k$ is positive semi-definite.

Here we consider vector valued Gaussian fields ξ_A whose covariance is the Greens function of a constant coefficient, anisotropic, elliptic, discrete difference operator $\mathcal{A} = \nabla^* A \nabla$ (plus higher order terms). Our main object of interest is the corresponding gradient Gaussian field $\nabla\xi_A$, i.e., we consider the σ-algebra generated by the gradients. They are referred to as massless field in the language of quantum field theory. Gradient fields appear naturally in discrete elasticity where the energy only depends on the distance between the atoms. The analysis of gradient Gaussian fields is difficult because they exhibit long range correlations only decaying critically as $\mathbb{E}(\nabla_i\xi^r(x)\nabla_j\xi^s(y)) \propto |x-y|^{-d}$. Finite range decompositions of gradient Gaussian fields are the basis of a multi-scale approach to control the correlation structure of the fields and avoid logarithmic divergences that appear in naive approaches.

Finite range decompositions of quadratic forms have appeared in different places in mathematics. Hainzl and Seiringer obtained decompositions of radially symmetric functions into weighted integrals over tent functions [14]. The first decomposition for a setting without radial symmetry was obtained for the discrete Laplacian by Brydges, Guadagni, and Mitter in [11]. Their results are based on averaging the Poisson kernel. Brydges and Talaczyck in [9] generalised this result to quite general elliptic operators on $\mathbb{R}^m$ that can be written as $\mathcal{A} = \mathcal{B}^*\mathcal{B}$. Adams, Kotecký, and Müller adapted this work in [1] to the discrete anisotropic setting. Their decomposition has the property that the kernels $\mathcal{C}_{A,k}$ are analytic function of the operator A. Later, Bauerschmidt gave a very general construction based on the finite propagation speed of the wave equation and functional calculus [3].

The goal of this work is to improve the regularity of the previous constructions. We show lower bounds for the previous decomposition and modify the construction such that we can control the decay behaviour of the kernels in Fourier space from above and below. This implies that the integration map $F \to \mathbb{E}(F(\cdot + \xi_{A,k}))$ is differentiable with respect to the matrix A uniformly in the size N of the torus. Our results hold for vector valued fields and we allow for higher order terms in the elliptic operator which corresponds to general quadratic finite range interaction. This allows us to handle, e.g., realistic models for discrete elasticity where next to nearest neighbour interactions are included. The construction is based on the Bauerschmidt decomposition in [3] but in a previous version of this project [12] we started from the construction in [1].

The main application of finite range decompositions is the renormalisation group approach to problems in statistical mechanics. Renormalisation was introduced by Wilson in the analysis of phase transitions [20]. Brydges and Yau [10] adapted Wilson's ideas to the statistical mechanics setting and initiated a long stream of developments. Recently Bauerschmidt, Brydges, and Slade introduced a general framework and investigate the

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