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Spectral enclosures for non-self-adjoint extensions of symmetric operators

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The spectral properties of non-self-adjoint extensions $A_{[B]}$ of a symmetric operator in a Hilbert space are studied with the help of ordinary and quasi boundary triples and the corresponding Weyl functions. These extensions are given in terms of abstract boundary conditions involving an (in general non-symmetric) boundary operator B. In the abstract part of this paper, sufficient conditions for sectoriality and m-sectoriality as well as sufficient conditions for $A_{[B]}$ to have a non-empty resolvent set are provided in terms of the parameter B and the Weyl function. Special attention is paid to Weyl functions that decay along the negative real line or inside some sector in the complex plane, and spectral enclosures for $A_{[B]}$ are proved in this situation. The abstract results are applied to elliptic differential operators with local and non-local Robin boundary conditions on unbounded domains, to Schrödinger operators with δ-potentials of complex strengths supported on unbounded hypersurfaces or

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infinitely many points on the real line, and to quantum graphs with non-self-adjoint vertex couplings.

Contents

1. Introduction

Spectral problems for differential operators in Hilbert spaces and related boundary value problems have attracted a lot of attention in the last decades and have strongly influenced the development of modern functional analysis and operator theory. For example, the classical treatment of Sturm–Liouville operators and the corresponding Titchmarsh–Weyl theory in Hilbert spaces have led to the abstract concept of boundary triples and their Weyl functions (see [43,55,82,96]), which is an efficient and well-established tool to investigate closed extensions of symmetric operators and their spectral properties via abstract boundary maps and an analytic function; see, e.g. [1,5,40–42,44, 53,56,115,117,125]. The more recent notion of quasi boundary triples and their Weyl functions are inspired by PDE analysis in a similar way. This abstract concept from [22, 24] is tailor-made for spectral problems involving elliptic partial differential operators and the corresponding boundary value problems; the Weyl function of a quasi boundary triple is the abstract counterpart of the Dirichlet-to-Neumann map. For different abstract treatments of elliptic PDEs and Dirichlet-to-Neumann maps we refer to the classical works [84,128] and the more recent approaches [11–13,30,54,77–80,83,91,118, 122,124].

To recall the notions of ordinary and quasi boundary triples in more detail, let S be a densely defined, closed, symmetric operator in a Hilbert space $(\mathcal{H},(\cdot,\cdot)_{\mathcal{H}})$ and let S^* denote its adjoint; then $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is said to be an *ordinary boundary triple* for S^* if $\Gamma_0,\Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$ are linear mappings from the domain of S^* into an auxiliary Hilbert space $(\mathcal{G},(\cdot,\cdot)_{\mathcal{G}})$ that satisfy the abstract Lagrange or Green identity