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Concentration for Coulomb gases and Coulomb transport inequalities [☆]

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ABSTRACT

We study the non-asymptotic behavior of Coulomb gases in dimension two and more. Such gases are modeled by an exchangeable Boltzmann–Gibbs measure with a singular two-body interaction. We obtain concentration of measure inequalities for the empirical distribution of such gases around their equilibrium measure, with respect to bounded Lipschitz and Wasserstein distances. This implies macroscopic as well as mesoscopic convergence in such distances. In particular, we improve the concentration inequalities known for the empirical spectral distribution of Ginibre random matrices. Our approach is remarkably simple and bypasses the use of renormalized energy. It crucially relies on new inequalities between probability metrics, including Coulomb transport inequalities which can be of independent interest. Our work is inspired by the one of Maïda and Maurel-Segala, itself inspired by large deviations techniques. Our approach allows to recover, extend, and simplify previous results by Rougerie and Serfaty.

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1. Introduction

The aim of this work is the non-asymptotic study of Coulomb gases around their equilibrium measure. We start by recalling some essential aspects of electrostatics, including the notion of equilibrium measure. We then incorporate randomness and present the Coulomb gas model followed by the natural question of concentration of measure for its empirical measure. This leads us to the study of inequalities between probability metrics, including new Coulomb transport type inequalities. We then state our results on concentration of measure and their applications and ramifications. We close the introduction with some additional notes and comments.

In all this work, we take $d \geq 2$.

1.1. Electrostatics

The d-dimensional *Coulomb kernel* is defined by

$$x \in \mathbb{R}^d \mapsto g(x) := \begin{cases} \log \dfrac{1}{|x|} & \text{if } d = 2, \\ \dfrac{1}{|x|^{d-2}} & \text{if } d \geq 3. \end{cases}$$

When $d = 3$, up to a multiplicative constant, $g(x)$ is the electrostatic potential at $x \in \mathbb{R}^3$ generated by a unit charge at the origin according to Coulomb's law. More generally, it is the fundamental solution of Poisson's equation. Namely, according to [31, Th. 6.20], if we denote by $\Delta := \partial_1^2 + \cdots + \partial_d^2$ the Laplace operator on $\mathbb{R}^d$ and by δ_0 the Dirac mass at the origin then, in the sense of Schwartz distributions,

$$\Delta g = -c_d \, \delta_0, \tag{1.1}$$

where c_d is a positive constant given by

$$c_d := \begin{cases} 2\pi & \text{if } d = 2, \\ (d-2)|\mathbb{S}^{d-1}| & \text{if } d \geq 3, \end{cases} \qquad \text{with} \qquad |\mathbb{S}^{d-1}| := \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

The function g is superharmonic on $\mathbb{R}^d$, harmonic on $\mathbb{R}^d \setminus \{0\}$, and belongs to the space $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ of locally Lebesgue-integrable functions.

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$. For any $\mu \in \mathcal{P}(\mathbb{R}^d)$ with compact support, its *Coulomb energy*,

$$\mathcal{E}(\mu) := \iint g(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \in \mathbb{R} \cup \{+\infty\}, \tag{1.2}$$

is well defined since g is bounded from below on any compact subset of $\mathbb{R}^d$ (actually we even have $g \geq 0$ when $d \geq 3$). Recall that a closed subset of $\mathbb{R}^d$ has *positive capacity*

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