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Quantitative minimality of strictly stable extremal submanifolds in a flat neighbourhood

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In this paper we extend the results of *A strong minimax property of nondegenerate minimal submanifolds*, by White, where it is proved that any smooth, compact submanifold, which is a strictly stable critical point for an elliptic parametric functional, is the unique minimizer in a certain geodesic tubular neighbourhood. We prove a similar result, replacing the tubular neighbourhood with one induced by the flat distance and we provide quantitative estimates. Our proof is based on the introduction of a penalized minimization problem, in the spirit of *A selection principle for the sharp quantitative isoperimetric inequality*, by Cicalese and Leonardi, which allows us to exploit the regularity theory for almost minimizers of elliptic parametric integrands.

1. Introduction

It is well known that any strictly stable critical point of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is locally its unique minimizer. In [10], B. White proves a statement of similar nature in a

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space of submanifolds of a Riemannian manifold, where the function f is replaced by an elliptic parametric functional. In his setting the term "locally" above should be intended with respect to the *strong* topology induced by the Riemannian distance. In the present paper we improve such result, replacing the strong topology with the one induced by the flat distance and providing also quantitative estimates. More precisely we prove the following result, where by $\mathbb{F}(T)$ we denote the flat norm of the integral current T.

1.1. Theorem. *Let M^m be a smooth, compact Riemannian manifold (or $M = \mathbb{R}^m$)*[1] *and suppose that $\Sigma^n \subset M^m$ is a smooth, embedded, compact, oriented submanifold with (possibly empty) boundary which is a strictly stable critical point for a smooth, elliptic parametric functional F. Then there exist $\varepsilon > 0$ and $C > 0$ (depending on Σ and M) such that*

$$F(S) \geq F(\Sigma) + C(\mathbb{F}(S - \Sigma))^2, \tag{1.1}$$

whenever $0 < \mathbb{F}(S - \Sigma) \leq \varepsilon$ and S is an integral current on M, homologous to Σ.

Following [10], in Theorem 5.1 we exploit the previous result to prove a minimax property of unstable, but nondegenerate minimal submanifolds.

1.2. Idea of the proof of Theorem 1.1. Here is a sketch of the proof of Theorem 1.1. For simplicity, we replace (1.1) with the weaker (non quantitative) inequality $F(S) > F(\Sigma)$, which would imply that Σ is uniquely minimizing in the flat neighbourhood. The proof is by contradiction, and it is inspired by the technique used in [2]. We assume that for every $\delta > 0$ we can select S_δ, homologous to Σ, which satisfies $F(S_\delta) \leq F(\Sigma)$ and $0 < \mathbb{F}(S_\delta - \Sigma) < \delta$. We denote $\eta_\delta := \mathbb{F}(S_\delta - \Sigma)$ and define, for $\lambda > 0$, a penalized functional $F_{\delta,\lambda}$ by

$$F_{\delta,\lambda}(T) := F(T) + \lambda|\mathbb{F}(T - \Sigma) - \eta_\delta| \,.$$

We then consider integral currents

$$R_{\delta,\lambda} \in \operatorname{argmin}\{F_{\delta,\lambda}(T) : T \text{ is homologous to } \Sigma\}.$$

By definition, we have

$$F(R_{\delta,\lambda}) \leq F_{\delta,\lambda}(R_{\delta,\lambda}) \leq F_{\delta,\lambda}(S_\delta) = F(S_\delta) \leq F(\Sigma)\,,$$

[1] More generally, it suffices to require that there exists an embedding of M into $\mathbb{R}^d$ and a tubular neighbourhood of M which admits a Lipschitz projection π onto M. Indeed, with such assumption it is possible to recast the problem in the Euclidean setting, via the machinery introduced in [9, §8] and the technique used in Lemma 3.4.