Journal of Functional Analysis 275 (2018) 1100–1137

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Noncommutative multi-parameter Wiener–Wintner type ergodic theorem

Guixiang Hong [a,b], Mu Sun [c,*]

[a] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*
[b] *Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China*
[c] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China*

ARTICLE INFO

Article history:
Received 2 June 2017
Accepted 18 May 2018
Available online 26 May 2018
Communicated by Stefaan Vaes

MSC:
primary 46L53, 46L55
secondary 47A35, 37A99

Keywords:
Noncommutative L_p-spaces
Noncommutative dynamical systems
Wiener–Wintner type ergodic theorem
Multi-parameter individual ergodic theorem

ABSTRACT

In this paper, we establish a multi-parameter version of Bellow and Losert's Wiener–Wintner type ergodic theorem for dynamical systems not necessarily commutative. More precisely, we introduce a weight class $\mathcal{D}$, which is shown to strictly include the multi-parameter bounded Besicovitch weight class, thus including the set

$$\Lambda_d = \left\{ \{\lambda_1^{k_1} \cdots \lambda_d^{k_d}\}_{(k_1,\ldots,k_d)\in\mathbb{N}^d} : \quad (\lambda_1,\ldots,\lambda_d) \in \mathbb{T}^d \right\};$$

then we prove a multi-parameter Bellow and Losert's Wiener–Wintner type ergodic theorem for the class $\mathcal{D}$ and for a noncommutative trace preserving dynamical system $(\mathcal{M}, \tau, \mathbf{T})$, $\mathcal{M}$ being a von Neumann algebra. Restricted to Λ_d, we also prove a noncommutative multi-parameter analogue of Bourgain's uniform Wiener–Wintner ergodic theorem.

The "noncommutativity" and the "multi-parameter" characters induce some difficulties in the proofs. For instance, our argument of proving the uniform convergence for a dense subset turns out to be quite different from the classical case since the "pointwise" argument does not work in the noncommutative setting; also to obtain the uniform convergence in the largest spaces, we need a maximal inequality between the Orlicz spaces, but it cannot be deduced by using classical

* Corresponding author.
E-mail addresses: guixiang.hong@whu.edu.cn (G. Hong), musun@hust.edu.cn (M. Sun).

https://doi.org/10.1016/j.jfa.2018.05.016

extrapolation argument directly. Junge and Xu's noncommutative maximal inequalities with the optimal order, together with the atomic decomposition of Orlicz spaces, play the essential role in overcoming the second difficulty.

1. Introduction

In classical ergodic theory, $(X, \mathcal{F}, \mu, T)$ is called a finite measure-preserving dynamical system if $(X, \mathcal{F}, \mu)$ is a finite measure space and T a measure preserving transformation on X. In 1941, Wiener and Wintner [38] showed that for any such dynamical system $(X, \mathcal{F}, \mu, T)$ and any $f \in L_1(\mu)$, there exists a set X_f of full measure in X such that for any $x \in X_f$, the sequence

$$\frac{1}{n+1} \sum_{k=0}^{n} \lambda^k f\left(T^k x\right)$$

converges for all $\lambda \in \mathbb{T}$ (the 1-dimensional complex torus).

This result proved by Wiener and Wintner is outstanding to some extent. Since intuitively the intersection of the sets of convergence $X_{f,\lambda}$ (from the application of the Dunford–Schwartz ergodic theorem associated with the contraction λT to the function f) written as $\cap_{\lambda \in \mathbb{T}} X_{f,\lambda}$ may be empty.

To describe further development of Wiener–Wintner's theorem, let us introduce the following notion. Let $\mathcal{B}(\mu)$ be a set of functions constructed from a finite measure space $(X, \mathcal{F}, \mu)$.

Definition 1.1. A set $\mathcal{A}$ of sequences of complex numbers $a = \{a(k)\}_{k=0}^{\infty}$ is called $\mathcal{B}$-Wiener–Wintner type (in short $\mathcal{B}$-WW type), if for any measure preserving system $(X, \mathcal{F}, \mu, T)$ and any $f \in \mathcal{B}(\mu)$, there exists X_f of full measure in X, such that for any $x \in X_f$, the sequence

$$\frac{1}{n+1} \sum_{k=0}^{n} a(k) f\left(T^k x\right)$$

converges for all $a \in \mathcal{A}$.

Using this definition, Wiener–Wintner's theorem can be reformulated as: *the set* $\Lambda_1 = \{(\lambda^k)_{k=0}^{\infty} : \lambda \in \mathbb{T}\}$ *is of* L_1*-WW type.* To provide a full description or characterization of the largest L_1-WW type set (or more generally $\mathcal{B}$-WW type set) becomes a natural and interesting direction. Two important advances have been made since the appearance