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On Gagliardo–Nirenberg type inequalities in Fourier–Herz spaces

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ABSTRACT

A variant of the Gagliardo–Nirenberg inequality in Hat–Sobolev spaces is proved, which improves certain classes of classical Sobolev embeddings. Some continuation criterion for the incompressible Navier–Stokes system is established as an application. A direct proof of the fractional Gagliardo–Nirenberg inequality in end-point Besov spaces is given and as a corollary, its counterpart in Fourier–Herz spaces is established.

1. Introduction and main results

In the original proofs given independently by E. Gagliardo [12] and L. Nirenberg [33], the celebrated Gagliardo–Nirenberg inequalities in the d-dimensional Euclidian space are derived by interpolating two functional inequalities. The first ingredient is a family of Sobolev inequalities whose proof nowadays can be found in various literature. The second one is a dimension-free, complex interpolation inequality, often called the con-

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vexity Hölder inequality. Once the two end-point inequalities are established, a standard interpolation argument in Lebesgue spaces yields the general case.

The fractional generalization of the Gagliardo–Nirenberg inequality has been considered by many. Among those, the most general form is presented by [18], without specifying the best constants. For the study of optimality of the constants of the Gagliardo–Nirenberg inequality in various situations, we refer to [36,7,8] and the references therein. Concerning works on weighted Gagliardo–Nirenberg inequalities or *Caffarelli–Kohn–Nirenberg inequalities* (first established in [4]) and optimal constants, we refer to [6], [10] and [9].

The purpose of this paper is to present a straightforward proof to the classical (fractional) Gagliardo–Nirenberg inequality as well as its counterpart in Hat–Sobolev and Fourier–Herz spaces in the spirit of [12,33]. The Gagliardo–Nirenberg type inequality in Hat–Sobolev spaces can be considered as refinements of the classical ones in a certain range of exponents.

We present the constants explicitly whenever they are relatively easy to compute. It is interesting that we may obtain a maximizer for the convexity Hölder inequality in Hat–Sobolev spaces for some special cases (Theorem 3.2). As an application of the Gagliardo–Nirenberg type inequality, we prove a variant of the blow-up criterion for the incompressible Navier–Stokes system.

1.1. Notation

Before going into our main result, let us prepare some notation. Hereafter, we denote by $L^p(\mathbb{R}^d)$ $(1 \le p \le \infty)$ the standard Lebesgue spaces on the d-dimensional Euclidean space $\mathbb{R}^d$, and by $\ell^p(\mathbb{Z})$ the set of sequences with summable p-th powers. Unless otherwise stated, we assume $d \ge 1$ throughout the paper. We abbreviate $\|\cdot\|_{L^p(\mathbb{R}^d)} = \|\cdot\|_{L^p}$ whenever the dimension d is not relevant. A C^∞ complex-valued function f on $\mathbb{R}^d$ is called a Schwartz function if for every pair of multi-indices α and β there exists a positive constant $C_{\alpha,\beta}$ such that $\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f| = C_{\alpha,\beta} < \infty$. We denote the set of all Schwartz functions on $\mathbb{R}^d$ by $\mathcal{S}(\mathbb{R}^d)$ and its topological dual by $\mathcal{S}'(\mathbb{R}^d)$. Elements in $\mathcal{S}'(\mathbb{R}^d)$ are called tempered distributions. We define the fractional derivative operator by

$$\Lambda^s := (-\Delta)^{s/2} = \mathcal{F}^{-1}|\cdot|^s\mathcal{F} \tag{1.1}$$

for $s \in \mathbb{R}$, where $\mathcal{F}u = \widehat{u}$ denotes the Fourier transform of $u \in \mathcal{S}'(\mathbb{R}^d)$.

Definition 1.1 *(Homogeneous (fractional) Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$).* Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. Let $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$ be the space of all polynomials over $\mathbb{R}^d$. We define the homogeneous Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$ by

$$\dot{H}^s_p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}\,;\ \|f\|_{\dot{H}^s_p(\mathbb{R}^d)} < \infty\}$$