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Spectral theory of multiplication operators on Hardy–Sobolev spaces ☆

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For a pointwise multiplier φ of the Hardy–Sobolev space H^2_β on the open unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, we study spectral properties of the multiplication operator $M_\varphi : H^2_\beta \to H^2_\beta$. In particular, we compute the spectrum and essential spectrum of M_φ and develop the Fredholm theory for these operators.

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1. Introduction

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$ and $H(\mathbb{B}_n)$ be the space of all holomorphic functions on $\mathbb{B}_n$. For $f \in H(\mathbb{B}_n)$ we use

$$Rf(z) = z_1 \frac{\partial f}{\partial z_1}(z) + \cdots + z_n \frac{\partial f}{\partial z_n}(z)$$

to denote the radial derivative of f at z. If

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

is the homogeneous expansion of f, then it is easy to see that

$$Rf(z) = \sum_{k=0}^{\infty} k f_k(z) = \sum_{k=1}^{\infty} k f_k(z).$$

More generally, for any real β and any $f \in H(\mathbb{B}_n)$ with the homogeneous expansion above, we define

$$R^\beta f(z) = \sum_{k=1}^{\infty} k^\beta f_k(z)$$

and call it the radial derivative of f of order β.

It is clear that these fractional radial differential operators satisfy $R^\alpha R^\beta = R^{\alpha+\beta}$. When $\beta < 0$, the effect of R^β on f is actually "integration" instead of "differentiation". For example, radial differentiation of order -3 is actually radial integration of order 3.

For $\beta \in \mathbb{R}$ the Hardy–Sobolev space H^2_β consists of all holomorphic functions f on $\mathbb{B}_n$ such that $R^\beta f$ belongs to the classical Hardy space H^2. It is clear that H^2_β is a Hilbert space with the inner product

$$\langle f, g\rangle_\beta = f(0)\overline{g(0)} + \langle R^\beta f, R^\beta g\rangle_{H^2}.$$

The induced norm in H^2_β is then given by

$$\|f\|^2_\beta = |f(0)|^2 + \|R^\beta f\|^2_{H^2}.$$

The multiplier algebra of H^2_β, denoted by $\mathcal{M}_\beta$, consists of all functions $\varphi \in H(\mathbb{B}_n)$ such that $\varphi f \in H^2_\beta$ for every $f \in H^2_\beta$. A standard application of the closed-graph theorem shows that every $\varphi \in \mathcal{M}_\beta$ induces a bounded linear operator $M_\varphi : H^2_\beta \to H^2_\beta$. The purpose of this paper is to study the spectral properties of these multiplication operators. Our main results are the following.