Journal of Functional Analysis 275 (2018) 1280–1319

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Sharpening Hölder's inequality

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ARTICLE INFO

Article history:
Received 10 October 2017
Accepted 7 May 2018
Available online 17 May 2018
Communicated by K. Seip

Keywords:
Hölder's inequality
Sharpening
Pythagorean theorem

ABSTRACT

We strengthen Hölder's inequality. The new family of sharp inequalities we obtain might be thought of as an analog of the Pythagorean theorem for the L^p-spaces. Our treatment of the subject matter is based on Bellman functions of four variables.

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[1] Support by RSF grant 14-41-00010.
[2] Support by RSF grant 14-21-00035.

https://doi.org/10.1016/j.jfa.2018.05.003
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1. Introduction

1.1. The Cauchy–Schwarz inequality and the Pythagorean theorem

Let $\mathcal{H}$ be a Hilbert space (over the complex or the reals) with an inner product $\langle\cdot,\cdot\rangle$. The Pythagorean theorem asserts

$$\left|\left\langle f, \frac{e}{\|e\|}\right\rangle\right|^2 + \|\mathbf{P}_{e^\perp} f\|^2 = \|f\|^2, \qquad e, f \in \mathcal{H}, \;\; e \neq 0. \tag{1.1}$$

Here, $\mathbf{P}_{e^\perp}$ denotes the orthogonal projection onto the orthogonal complement of a non-trivial vector e. At this point, we note that since $\|\mathbf{P}_{e^\perp} f\| \geqslant 0$, the identity (1.1) implies the Cauchy–Schwarz inequality

$$|\langle f, e\rangle| \leqslant \|f\|\,\|e\|, \qquad e, f \in \mathcal{H}.$$

We also note that (1.1) leads to Bessel's inequality:

$$\sum_{n=1}^{N} |\langle f, e_n\rangle|^2 \leqslant \|f\|^2, \qquad f \in \mathcal{H},$$

for an orthonormal system $e_1, \ldots, e_N$ in $\mathcal{H}$.

We may think of (1.1) as of an expression of the precise loss in the Cauchy–Schwarz inequality. Our aim in this paper is to find an analogous improvement for the well-known Hölder inequality for L^p norms. Before we turn to the analysis of L^p spaces, we need to replace the norm of the projection, $\|\mathbf{P}_{e^\perp} f\|$, by an expression which does not rely on the Hilbert space structure. It is well known that

$$\|\mathbf{P}_{e^\perp} f\| = \inf_{\alpha} \|f - \alpha e\|, \tag{1.2}$$

where α ranges over all scalars (real or complex).

1.2. Background on Hölder's inequality for L^θ

We now consider $L^\theta(X,\mu)$, where (X,μ) is a standard σ-finite measure space. We sometimes focus our attention on finite measures, but typically the transfer to the more general σ-finite case is an easy exercise. The functions are assumed complex valued. Throughout the paper we assume the summability exponents are in the interval $(1,+\infty)$, in particular, $1 < \theta < +\infty$. We reserve the symbol p for the range $[2,\infty)$ and q for $(1,2]$ (we also usually assume that p and q are dual in the sense $\frac{1}{p} + \frac{1}{q} = 1$). Also, to simplify the presentation, we assume μ has no atoms.

Our point of departure is Hölder's inequality, which asserts that in terms of the sesquilinear form

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