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A dichotomy property for locally compact groups ☆

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We extend to metrizable locally compact groups Rosenthal's theorem describing those Banach spaces containing no copy of ℓ_1. For that purpose, we transfer to general locally compact groups the notion of interpolation (I_0) set, which was defined by Hartman and Ryll-Nardzewsky [24] for locally compact abelian groups. Thus we prove that for every sequence $\{g_n\}_{n<\omega}$ in a locally compact group G, then either $\{g_n\}_{n<\omega}$ has a weak Cauchy subsequence or contains a subsequence that is an I_0 set. This result is subsequently applied to obtain sufficient conditions for the existence of Sidon sets in a locally compact group G, an old question that remains open since 1974 (see [31] and [19]). Finally, we show that every locally compact group strongly respects compactness extending thereby a result by Comfort, Trigos-Arrieta, and Wu [13], who established this property for abelian locally compact groups.

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1. Introduction

A well-known result of Rosenthal establishes that if $\{x_n\}_{n<\omega}$ is a bounded sequence in a Banach space X, then either $\{x_n\}_{n<\omega}$ has a weak Cauchy subsequence, or $\{x_n\}_{n<\omega}$ has a subsequence equivalent to the usual ℓ_1-basis (and then X contains a copy of ℓ_1). In this paper, we look at this result for locally compact groups. More precisely, our main goal is to extend Rosenthal's dichotomy theorem on Banach spaces to locally compact groups and their weak topologies. First, we need some definitions and basic results.

Given a locally compact group (G, τ), we denote by $Irr(G)$ the set of all continuous unitary irreducible representations σ defined on G. That is, continuous in the sense that each matrix coefficient function $g \mapsto \langle \sigma(g)u, v\rangle$ is a continuous map of G into the complex plane. Thus, fixed $\sigma \in Irr(G)$, if $\mathcal{H}^\sigma$ denotes the Hilbert space associated to σ, we equip the unitary group $\mathbb{U}(\mathcal{H}^\sigma)$ with the weak (equivalently, strong) operator topology. For two elements π and σ of $Irr(G)$, we write $\pi \sim \sigma$ to denote the relation of unitary equivalence and we denote by $\widehat{G}$ the *dual object* of G, which is defined as the set of equivalence classes in $(Irr(G)/\sim)$. We refer to [14,4] for all undefined notions concerning the unitary representations of locally compact groups.

Adopting, the terminology introduced by Ernest in [16], set $\mathcal{H}_n \overset{\text{def}}{=} \mathbb{C}^n$ for $n = 1, 2, \ldots$; and $\mathcal{H}_0 \overset{\text{def}}{=} l^2(\mathbb{Z})$. The symbol $Irr_n^C(G)$ will denote the set of irreducible unitary representations of G on $\mathcal{H}_n$, where it is assumed that every set $Irr_n^C(G)$ is equipped with the compact open topology. Finally, define $Irr^C(G) = \bigsqcup_{n\geq 0} Irr_n^C(G)$ (the disjoint topological sum).

We denote by $G^w = (G, w(G, Irr(G))$ (resp. $G^{w_C} = (G, w(G, Irr^C(G))))$ the group G equipped with the weak (group) topology generated by $Irr(G)$ (resp. $Irr^C(G)$). Since equivalent representations define the same topology, we have $G^w = (G, w(G, \widehat{G}))$. That is, the *weak topology* is the initial topology on G defined by the dual object. Moreover, in case G is a separable, metric, locally compact group, then every irreducible unitary representation acts on a separable Hilbert space and, as a consequence, is unitary equivalent to a member of $Irr^C(G)$. Thus $G^w = (G, w(G, Irr^C(G))) = G^{w_C}$ for separable, metric, locally compact groups. We will make use of this fact in order to avoid the proliferation of isometries (see [14]). In case the group G is abelian, the dual object $\widehat{G}$ is a group, which is called *dual group*, and the weak topology of G reduces to the weak topology generated by all continuous homomorphisms of G into the unit circle $\mathbb{T}$. That is, the weak topology coincides with the so-called *Bohr topology* of G, that we recall next for the reader's sake.

With every (not necessarily abelian) topological group G there is associated a compact Hausdorff group bG, the so-called *Bohr compactification* of G, and a continuous homomorphism b of G onto a dense subgroup of bG such that bG is characterized by the following universal property: given any continuous homomorphism h of G into a compact group K, there is always a continuous homomorphism $\bar{h}$ of bG into K such that $h = \bar{h} \circ b$ (see [28, V §4], where a detailed study on bG and their properties is given). In Anzai