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HILBERT SPACE OPERATORS WITH COMPATIBLE OFF-DIAGONAL CORNERS

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ABSTRACT. Given a complex, separable Hilbert space $\mathcal{H}$, we characterize those operators for which $\|PT(I-P)\| = \|(I-P)TP\|$ for all orthogonal projections P on $\mathcal{H}$. When $\mathcal{H}$ is finite-dimensional, we also obtain a complete characterization of those operators for which $\operatorname{rank}(I-P)TP = \operatorname{rank} PT(I-P)$ for all orthogonal projections P. When $\mathcal{H}$ is infinite-dimensional, we show that any operator with the latter property is normal, and its spectrum is contained in either a line or a circle in the complex plane.

1. INTRODUCTION

1.1. Let $\mathcal{H}$ be a complex, separable Hilbert space. By $\mathcal{B}(\mathcal{H})$, we denote the algebra of bounded, linear operators on $\mathcal{H}$. If $\dim \mathcal{H} = n < \infty$, then we identify $\mathcal{H}$ with $\mathbb{C}^n$ and $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_n(\mathbb{C})$.

One of the most important open problems in operator theory is the *Invariant Subspace Problem*, which asks whether or not every bounded, linear operator T acting on a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$ admits a non-trivial invariant subspace; that is, a closed subspace $\mathcal{M} \notin \{\{0\}, \mathcal{H}\}$ for which $T\mathcal{M} \subseteq \mathcal{M}$.

We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is **(orthogonally) reductive** if for each orthogonal projection $P \in \mathcal{B}(\mathcal{H})$, the condition $PT(I-P) = 0$ implies that $(I-P)TP = 0$. The *Reductive Operator Conjecture* is the assertion that every reductive operator is normal. It was shown by Dyer, Pederson and Procelli [9] that the Invariant Subspace Problem admits a positive solution if and only if the *Reductive Operator Conjecture* is true.

Our goal in this paper is to study two variants of orthogonal reductivity. Let $T \in \mathcal{B}(\mathcal{H})$ and $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection. We refer to the operator

$$P^\perp TP : P\mathcal{H} \to P^\perp\mathcal{H}$$

as an **off-diagonal corner** of T.

Relative to the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$, we may write $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. We refer to the block-entries of such block-matrices via their geographic positions: NW, NE, SE, SW, and the NE and the SW block-entries are examples of the off-diagonal corners.

In the work below, we shall be interested in two phenomena: firstly, when the operator norm of $B(= B_P)$ coincides with the operator norm of $C(= C_P)$ for all projections P, and secondly, when the rank of B coincides with the rank of C for all projections P. Clearly, any operator which satisfies one of these two conditions is orthogonally reductive. An example is given in Section 5 below to show that the converse to this statement is false.

In the case of normal matrices, some related work has been done by Bhatia and Choi [5]. For instance, if the dimension of the space is $2n < \infty$, and if P is a projection of rank n, it is a consequence of the fact that the Euclidean norm of the k^{th} column of a normal matrix coincides with that of the k^{th} row for all k that the Hilbert-Schmidt (or Frobenius) norm of B always equals that of C. Further, they show that $\|B\| \leq \sqrt{n}\|C\|$, and that equality can be achieved for some normal matrix $T \in \mathbb{M}_{2n}(\mathbb{C})$ and some projection P of rank n if and only if $n \leq 3$.

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