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PII: S0022-1236(17)30420-2
DOI: https://doi.org/10.1016/j.jfa.2017.11.002
Reference: YJFAN 7912

To appear in: *Journal of Functional Analysis*

Received date: 26 September 2017
Accepted date: 8 November 2017

A HELSON MATRIX WITH EXPLICIT EIGENVALUE ASYMPTOTICS

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ABSTRACT. A Helson matrix (also known as a multiplicative Hankel matrix) is an infinite matrix with entries $\{a(jk)\}$ for $j, k \geq 1$. Here the (j,k)'th term depends on the product jk. We study a self-adjoint Helson matrix for a particular sequence $a(j) = (\sqrt{j}\log j(\log\log j)^\alpha))^{-1}$, $j \geq 3$, where $\alpha > 0$, and prove that it is compact and that its eigenvalues obey the asymptotics $\lambda_n \sim \varkappa(\alpha)/n^\alpha$ as $n \to \infty$, with an explicit constant $\varkappa(\alpha)$. We also establish some intermediate results (of an independent interest) which give a connection between the spectral properties of a Helson matrix and those of its continuous analogue, which we call the integral Helson operator.

1. Introduction

1.1. **Background: Hankel matrices.** We start our discussion by recalling relevant facts from the theory of Hankel matrices. Let $\{b(j)\}_{j=0}^\infty$ be a sequence of complex numbers. A *Hankel matrix* is an infinite matrix of the form

$$H(b) = \{b(j+k)\}_{j,k=0}^\infty,$$

considered as a linear operator in $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. One of the key examples of Hankel matrices is the *Hilbert matrix*, which corresponds to the choice $b(j) = 1/(j+1)$. It is well known that the Hilbert matrix is bounded (but not compact). From the boundedness of the Hilbert matrix by a simple argument one obtains

$$b(j) = o(1/j), \quad j \to \infty \quad \Rightarrow \quad H(b) \text{ is compact.}$$

A natural family of compact self-adjoint Hankel operators of this class was considered in [13]. To state this result, we need some notation. For a compact self-adjoint operator A, let us denote by $\{\lambda_n^+(A)\}_{n=1}^\infty$ the non-increasing sequence of positive eigenvalues (enumerated with multiplicities taken into account), and let $\lambda_n^-(A) = \lambda_n^+(-A)$.

Theorem A. [13] *Let $b(j)$ be a sequence of real numbers defined by*

$$b(j) = 1/(j(\log j)^\alpha), \quad j \geq 2;$$

Date: November 14, 2017.
2010 *Mathematics Subject Classification.* 47B32, 47B35.
Key words and phrases. Hankel matrix, Helson matrix, spectral asymptotics, Schatten class.