Descriptive complexity of some isomorphism classes of Banach spaces

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DESCRIPTIVE COMPLEXITY OF SOME ISOMORPHISM CLASSES OF BANACH SPACES.

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Abstract. We present a topological frame in which it is possible to estimate the complexity of some Borel families of separable Banach spaces. We use this frame for evaluating the complexity of the isomorphism class of the Hilbert space ℓ_2, of certain asymptotically Hilbertian spaces, and of the ℓ_p spaces under the condition $1 < p < \infty$. The complexity appears to increase with the distance to the Hilbert space. Commented open problems conclude the article.

1. Introduction

The purpose of this work is to evaluate the descriptive complexity of some classes of real separable Banach spaces. Of course, the collection of separable Banach spaces is not a set, and we have first to code them with the elements of some Polish space. It turns out that when such a coding is performed, the equivalence relation of isometry between Banach spaces is not smooth (see [23] for a stronger result), and thus there is no hope of finding a coding which would map isometric spaces to the same code. Therefore we have to consider classes $\mathcal{C}$ of Banach spaces which are closed for the isometry relation: that is, if $X \in \mathcal{C}$ and Y is isometric to X, then $Y \in \mathcal{C}$. It is clear that any meaningful property of Banach spaces will provide such a class $\mathcal{C}$.

We recall that a topological space is Polish if it is homeomorphic to a separable metric complete space. We denote by $\Delta = 2^\omega$ the Cantor set. The separable Banach space $\mathcal{C}(\Delta)$ of real-valued continuous functions on Δ is isometrically universal for separable Banach spaces. We choose therefore to code the separable Banach spaces with the set SB of closed vector subspaces of $\mathcal{C}(\Delta)$. For providing the set SB with a topology, we first equip the set $\mathcal{F}(\mathcal{C}(\Delta))$ of non-empty closed subsets of $\mathcal{C}(\Delta)$ with an admissible Polish topology, keeping in mind that such a topology is not canonical. However, the Borel σ-field is the same for all admissible topologies (it is called the Effros-Borel structure) and the Borel class of a subset of $\mathcal{F}(\mathcal{C}(\Delta))$ is essentially independent of the choice of the admissible topology (see section 2). The subset SB of $\mathcal{F}(\mathcal{C}(\Delta))$ is shown to be a Polish subset of $\mathcal{F}(\mathcal{C}(\Delta))$ for any admissible topology (see Section 3). The frame is now ready for estimating the complexity of the isomorphism classes of some Banach spaces (see Section 4). After proving the existence of a continuous selection of a dense sequence in elements of SB, we show that the isomorphism class of the Hilbert space is a Borel set of class 2, that the isomorphism classes of certain *near Hilbert* spaces are Borel sets of class 4, and

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