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On a combinatorial curvature for surfaces with inversive distance circle packing metrics

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ABSTRACT

In this paper, we introduce a new combinatorial curvature on triangulated surfaces with inversive distance circle packing metrics. Then we prove that this combinatorial curvature has global rigidity. To study the Yamabe problem of the new curvature, we introduce a combinatorial Ricci flow, along which the curvature evolves almost in the same way as that of scalar curvature along the surface Ricci flow obtained by Hamilton [20]. Then we study the long time behavior of the combinatorial Ricci flow and obtain that the existence of a constant curvature metric is equivalent to the convergence of the flow on triangulated surfaces with nonpositive Euler number. We further generalize the combinatorial curvature to α-curvature and prove that it is also globally rigid, which is in fact a generalized Bowers–Stephenson conjecture [6]. We also use the combinatorial Ricci flow to study the corresponding α-Yamabe problem.

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1. Introduction

This is a continuation of our work on combinatorial curvature in [16]. This paper generalizes our results in [16] to triangulated surfaces with inversive distance circle packing metrics. Circle packing is a powerful tool in the study of differential geometry and geometric topology and there are lots of research on this topic. In his work on constructing hyperbolic structure on 3-manifolds, Thurston ([27], Chapter 13) introduced the notion of Euclidean and hyperbolic circle packing metrics on triangulated surfaces with prescribed intersection angles. The requirement of prescribed intersection angles corresponds to the fact that the intersection angle of two circles is invariant under the Möbius transformations. For triangulated surfaces with Thurston's circle packing metrics, there will be singularities at the vertices. The classical combinatorial Gauss curvature K_i is introduced to describe the singularity at the vertex v_i, which is defined as the angle deficit at v_i. Thurston's work generalized Andreev's work on circle packing metrics on a sphere [1,2]. Andreev and Thurston's work together gave a complete characterization of the space of the classical combinatorial Gauss curvature. As a corollary, they got the combinatorial-topological obstacle for the existence of a constant curvature circle packing metric, which could be written as combinatorial-topological inequalities. Chow and Luo [7] first introduced a combinatorial Ricci flow, a combinatorial analogue of the smooth surface Ricci flow, for triangulated surfaces with Thurston's circle packing metrics and got the equivalence between the existence of a constant curvature metric and the convergence of the combinatorial Ricci flow. This work is the cornerstone of applications of combinatorial surface Ricci flow in engineering up to now, see for example [29,31] and the references therein. Luo [22] once introduced a combinatorial Yamabe flow on triangulated surfaces with piecewise linear metrics to study the corresponding constant curvature problem. The combinatorial surface Ricci flow in [7] and the combinatorial Yamabe flow in [22] are recently written in a unified form in [30]. The first author [9, 10] introduced a combinatorial Calabi flow on triangulated surfaces with Thurston's Euclidean circle packing metrics and proved the equivalence between the existence of constant circle packing metric and the convergence of the combinatorial Calabi flow. The authors [14] further generalized the combinatorial Calabi flow to hyperbolic circle packing metrics and got similar results.

However, there are some disadvantages for the classical discrete Gauss curvature as stated in [16]. The first is that the classical Euclidean discrete Gauss curvature is invariant under scaling, i.e., $K_i(\lambda r) = K_i(r)$ for any positive constant λ. The second is that the classical discrete Gauss curvature tends to zero, not the Gauss curvature of the smooth surface, as triangulated surfaces approximate a smooth surface. Motivated by the two disadvantages, the authors [16] introduced a new combinatorial curvature defined as $R_i = \frac{K_i}{r_i^2}$ for triangulated surfaces with Thurston's Euclidean circle packing metrics. If we take $g_i = r_i^2$ as the analogue of the Riemannian metric, then we have $R_i(\lambda g) = \lambda^{-1} R_i(g)$, which has the same form as that of the smooth Gauss curvature. Furthermore, there are examples showing that this curvature actually approximates the smooth Gauss