Accepted Manuscript

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PII: S0022-1236(18)30182-4
DOI: https://doi.org/10.1016/j.jfa.2018.04.007
Reference: YJFAN 8011

To appear in: *Journal of Functional Analysis*

Received date: 14 May 2017
Accepted date: 23 April 2018

NEAR-INFINITY CONCENTRATED NORMS AND THE FIXED POINT PROPERTY FOR NONEXPANSIVE MAPS ON CLOSED, BOUNDED, CONVEX SETS

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Abstract. In this paper we define the concept of a near-infinity concentrated norm on a Banach space X with a boundedly complete Schauder basis. When $\|\cdot\|$ is such a norm, we prove that $(X, \|\cdot\|)$ has the fixed point property (FPP); that is, every nonexpansive self-mapping defined on a closed, bounded, convex subset has a fixed point. In particular, P.K. Lin's norm in ℓ_1 [14] and the norm $\nu_p(\cdot)$ (with $p = (p_n)$ and $\lim_n p_n = 1$) introduced in [3] are examples of near-infinity concentrated norms. When $\nu_p(\cdot)$ is equivalent to the ℓ_1-norm, it was an open problem as to whether $(\ell_1, \nu_p(\cdot))$ had the FPP. We prove that the norm $\nu_p(\cdot)$ always generates a nonreflexive Banach space $X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \dots))$ satisfying the FPP, regardless of whether $\nu_p(\cdot)$ is equivalent to the ℓ_1-norm. We also obtain some stability results.

1. Introduction and Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and C a subset of X. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. The Banach space X endowed with the norm $\|\cdot\|$ has the fixed point property (FPP) if every nonexpansive mapping defined from a closed bounded convex subset C of X into itself has a fixed point. This property is not preserved by isomorphism, that is, it strongly depends on the underlying norm [14] . There is a wide literature relating geometric properties of reflexive Banach spaces with the fulfilment of the fixed point property (see, for instance, the monographs [9], [13] and the references therein).

The Banach space ℓ_1 endowed with its standard norm $\|\cdot\|_1$ is a classical example of a nonreflexive Banach space that fails to have the FPP. It is possible to "perturb" this $(\ell_1, \|\cdot\|_1)$-example to obtain other Banach spaces that fail to have the FPP. One such class of Banach spaces are those that contain asymptotically isometric copies of ℓ_1.

Recall that a Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy (a.i.c.) of ℓ_1 if there are a sequence $(x_n) \subset X$ and a decreasing sequence $(\epsilon_n) \subset (0, 1)$ with $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty} (1 - \epsilon_n)|t_n| \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le \sum_{n=1}^{\infty} |t_n|$$

1991 *Mathematics Subject Classification.* 46B03, 47H09, 47H10.

Key words and phrases. fixed point property; nonexpansive mappings; renorming theory.

The third author is partially funded by CONACYT grants 243722 and 613207. The fourth author is partially supported by MCIN, Grant MTM2015-65242-C2-1-P and Andalusian Regional Government Grant FQM-127. She is also grateful to the Department of Mathematics of the University of Pittsburgh for its support while preparing this manuscript. The fifth author thanks the University of Newcastle for its financial support during part of the preparation of this paper; and Brailey, Gail and the Mathematics Department for their hospitality.