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A new index theorem for monomial ideals by resolutions

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ABSTRACT

We prove an index theorem for the quotient module of a monomial ideal. We obtain this result by resolving the monomial ideal by a sequence of essentially normal Hilbert modules, each of which is a direct sum of (weighted) Bergman spaces on balls.

1. Introduction

Let $\mathbb{B}^m$ be the unit ball in the complex space $\mathbb{C}^m$, and $L_a^2(\mathbb{B}^m)$ be the Bergman space of square integrable holomorphic functions on $\mathbb{B}^m$, and A be the algebra $\mathbb{C}[z_1, \cdots, z_m]$ of polynomials of m variables. The algebra A plays two roles in our study: one is that A

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[1] Recently passed away.

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is a dense subspace of the Hilbert space $L_a^2(\mathbb{B}^m)$, the other is that A acts on $L_a^2(\mathbb{B}^m)$ by Toeplitz operators.

In this article we are interested in an ideal I of A generated by monomials. Let $\overline{I}$ be the closure of I in $L_a^2(\mathbb{B}^m)$, and Q_I be the quotient Hilbert space $L_a^2(\mathbb{B}^m)/\overline{I}$. The first author proved in [5, Theorem 2.1] that the Toeplitz operators T_{z_i}, $i = 1, \ldots, m$, on $\overline{I}$, and the quotient Q_I are essentially normal,[2] i.e. the following commutators are compact

$$[T_{z_i}|_{\overline{I}}, \big(T_{z_j}|_{\overline{I}}\big)^*] \in \mathcal{K}(\overline{I}), \text{ and } [T_{z_i}|_{Q_I}, \big(T_{z_j}|_{Q_I}\big)^*] \in \mathcal{K}(Q_I), \ i, j = 1, \cdots, m.$$

Let $\mathfrak{T}(Q_I)$ be the unital C^*-algebra generated by the Toeplitz operators $T_{z_i}|_{Q_I}$, $i = 1, \cdots, m$. The above essential normality property of the Toeplitz operators gives the following extension sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{T}(Q_I) \longrightarrow C(\sigma_I^e) \longrightarrow 0,$$

where σ_I^e is the essential spectrum of the quotient tuple $(T_{z_1}|_{Q_I}, \cdots, T_{z_k}|_{Q_I})$ on Q_I, and $\mathcal{K}$ is the algebra of compact operators. By the Gelfand–Naimark theorem, σ_I^e is the spectrum space of the commutative C^*-algebra $\mathfrak{T}(Q_I)/\mathcal{K}$. Abusing the notion, we will sometimes refer to σ_I^e as the essential spectrum space of the algebra $\mathfrak{T}(Q_I)$. The index problem we want to answer in this article is to provide a good description of the above K-homology class.

The main difficulty in answering the question above is that the ideal I in general fails to be radical. This makes the geometric ideas introduced in [8] and [10] impossible to apply directly. The seed of the main idea in this article is the following observation discussed in [8, Section 5.2]. For $m = 2$, consider the ideal $I = \langle z_1^2 \rangle \subset A = \mathbb{C}[z_1, z_2]$. The quotient Q_I can be written as the sum of two space

$$L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D}),$$

where $\mathbb{D}$ is the unit disk inside the complex plane $\mathbb{C}$, and $L_{a,1}^2(-)$ (and $L_{a,2}^2(-)$) is the weighted Bergman space with respect to the weight defined by the defining function $1 - |z|^2$ (and $(1 - |z|^2)^2$). Define the restriction map $R_I : L_a^2(\mathbb{B}^2) \to L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D})$ by

$$R_I(f) := (f|_{z_1=0}, \frac{\partial f}{\partial z_1}|_{z_1=0}).$$

It is not hard to introduce a Hilbert $A = \mathbb{C}[z_1, z_2]$-module structure on $L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D})$ so that the following exact sequence of Hilbert modules holds,

$$0 \longrightarrow \overline{I} \longrightarrow L_a^2(\mathbb{B}^2) \longrightarrow L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D}) \longrightarrow 0.$$

[2] Arveson [1, Corollary 2.2] proved the similar result on the Drury–Arveson space.