Journal of Functional Analysis 275 (2018) 259–287

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Perturbations of self-adjoint operators in semifinite von Neumann algebras: Kato–Rosenblum theorem

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A R T I C L E   I N F O

Article history:
Received 3 July 2017
Accepted 17 April 2018
Available online 21 April 2018
Communicated by Dan Voiculescu

MSC:
primary 47C15
secondary 47L60, 47L20

Keywords:
The generalized wave operators
The Kato–Rosenblum theorem
Norm-ideal perturbations
von Neumann algebras

A B S T R A C T

In the paper, we prove an analogue of the Kato–Rosenblum theorem in a semifinite von Neumann algebra. Let $\mathcal{M}$ be a countably decomposable, properly infinite, semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let τ be a faithful normal semifinite tracial weight of $\mathcal{M}$. Suppose that H and H_1 are self-adjoint operators affiliated with $\mathcal{M}$. We show that if $H - H_1$ is in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$, then the *norm* absolutely continuous parts of H and H_1 are unitarily equivalent. This implies that the real part of a non-normal hyponormal operator in $\mathcal{M}$ is not a perturbation by $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ of a diagonal operator.

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[1] The first author was partly supported by NSFC (Grant No. 11671133).
[2] The third author was partly supported by NSFC (Grant No. 11401071) and the Fundamental Research Funds for the Central Universities (Grant No. DUT18LK23).
[3] The fourth author was partly supported by NSFC (Grant No. 11371222 and No. 11671133).

https://doi.org/10.1016/j.jfa.2018.04.006

1. Introduction

This paper is a continuation of the investigation, which we began in [10], of diagonalizations of self-adjoint operators modulo norm ideals in semifinite von Neumann algebras.

Let $\mathcal{K}$ be a complex separable infinite dimensional Hilbert space. Denote by $\mathcal{B}(\mathcal{K})$ the set of bounded linear operators on $\mathcal{K}$. The Weyl–von Neummann theorem [22,21] states that a self-adjoint operator in $\mathcal{B}(\mathcal{K})$ is a sum of a diagonal operator and an arbitrarily small Hilbert–Schmidt operator. A result by Kuroda in [19] implies that a self-adjoint operator in $\mathcal{B}(\mathcal{K})$ is a sum of a diagonal operator and an arbitrarily small Schatten p-class operator with $p > 1$. Berg and Sikonia independently showed in [2] and [17] that a normal operator in $\mathcal{B}(\mathcal{K})$ is a compact perturbation of a diagonal operator. In [20], Voiculescu proved a surprising result by showing that a normal operator in $\mathcal{B}(\mathcal{K})$ is a diagonal operator plus an arbitrarily small Hilbert–Schmidt operator. This result of Voiculescu has recently been generalized in [10] to semifinite von Neumann algebras with separable predual. It is worth noting that Kuroda's result in [19] was also extended to countably decomposable, properly infinite, semifinite von Neumann algebras in [10].

In the case of perturbations by trace class operators, the influential Kato–Rosenblum theorem (see [6] and [16]) provides an obstruction to diagonalizations, modulo the trace class, of self-adjoint operators in $\mathcal{B}(\mathcal{K})$. More specifically, if H and H_1 are densely defined self-adjoint operators on $\mathcal{K}$ such that $H - H_1$ is in the trace class, then the Kato–Rosenblum theorem asserts that the absolutely continuous parts of H and H_1 are unitarily equivalent. Thus, if a self-adjoint operator H in $\mathcal{B}(\mathcal{K})$ has a nonzero absolutely continuous spectrum, then H can not be a sum of a diagonal operator and a trace class operator.

The purpose of this paper is to provide a version of the Kato–Rosenblum theorem in a semifinite von Neumann algebra. (For general knowledge about von Neumann algebras, the reader is referred to [4,5,18].) Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space and let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a countably decomposable, properly infinite von Neumann algebra with a faithful normal tracial weight τ. A quick example (see Example 2.4.2) shows the existence of a self-adjoint operator A in $\mathcal{M}$, where A has a nonzero absolutely continuous spectrum and A is also a sum of a diagonal operator and an arbitrarily small operator in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$. Thus it is natural to explore suitable analogues of the Kato–Rosenblum theorem still holding in a general semifinite von Neumann algebra $\mathcal{M}$. In the current paper, we prove an generalization of the Kato–Rosenblum theorem in semifinite von Neumann algebras. It is also worth noting that Carey and Pincus [3] characterized a necessary and sufficient condition for the unitary equivalence of two bounded self-adjoint operators modulo the trace class, which is based on the Kato–Rosenblum theorem. Thus, for possible extensions, we are also working on an analogue of the Carey–Pincus theorem in semifinite von Neumann algebras.

Before stating the results of the paper, we recall the following notation. Let $(\mathscr{X}, \|\cdot\|)$ be a Banach space. A mapping $f : \mathbb{R} \to \mathscr{X}$ is *locally absolutely continuous* if, for all