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# Tensor tomography in periodic slabs

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ABSTRACT

The X-ray transform on the periodic slab $[0,1]\times\mathbb{T}^n$, $n \geq 0$, has a non-trivial kernel due to the symmetry of the manifold and presence of trapped geodesics. For tensor fields gauge freedom increases the kernel further, and the X-ray transform is not solenoidally injective unless $n = 0$. We characterize the kernel of the geodesic X-ray transform for $L^2$-regular $m$-tensors for any $m \geq 0$. The characterization extends to more general manifolds, twisted slabs, including the Möbius strip as the simplest example.



## 1. Introduction

We study geodesic X-ray tomography of tensor fields on the manifold $M = [0,1]\times\mathbb{T}^n$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $n \geq 0$. This class of manifolds includes the interval $[0,1]$ and the

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strip $[0,1] \times \mathbb{T}^1$. We only consider geodesics joining boundary points, excluding trapped geodesics.

Our main result is Theorem 2 which completely characterizes the kernel of the X-ray transform on $M$ for tensor fields of any order. The kernel is the sum of two kinds of functions: those arising from potentials (symmetrized differentials of tensor fields of lower order) and those depending only on the variable on $[0,1]$. See section 2 for details. That is, both gauge freedom and symmetry cause kernel. We are not aware of earlier observations — and, in particular, characterizations — of this kind of kernel. In particular, the X-ray transform is always non-injective, even for scalar fields in all dimensions.

It is easy to see that the same kernel is present in the infinite slab $[0,1] \times \mathbb{R}^n$, but we do not pursue characterizing the kernel in that case. Our result can be seen as the case for periodic $L^2_{\text{loc}}$ tensor fields on $[0,1] \times \mathbb{R}^n$. In practical problems where the slab is large but finite, we expect there to be a significant instability corresponding to the kernel of the infinite case.

Observe that if any decay or integrability conditions are imposed on scalar functions on $[0,1] \times \mathbb{R}^n$, then the obvious kernel vanishes. The X-ray transform can be easily seen to be injective on compactly supported functions in $[0,1] \times \mathbb{R}^n$ using Helgason's support theorem for all straight lines avoiding the line $[0,1] \times \{0\}$. A version of the support theorem for compactly supported $L^1$ functions can be obtained through mollification, see e.g. [6, Proposition 5].

The result can also be extended to broken ray tomography on the slab, where one of the surfaces of $M$ reflects rays. This can be achieved with a simple reflection argument; cf. [7,5,4].

Our result can also be extended to a broader class of manifolds. The slab $M$ can be obtained by identifying some opposite faces of $[0,1]^{1+n}$. If the gluing is done in a more exotic way, one ends up with what we call a twisted slab. The simplest example of a twisted slab is the Möbius strip. For more details, see Theorem 11 and section 3.

**Remark 1.** The periodic slab can be stretched in different ways. The interval can be any $[0,L]$ and we can divide $\mathbb{R}^n$ by any lattice obtained from $\mathbb{Z}^n$ by a linear bijection. For simplicity we restrict ourselves to $L = 1$ and the lattice $\mathbb{Z}^n$. The results can be generalized in an obvious way. If the lattice is stretched differently in different directions, then there are fewer twisted slabs. For the sake of clarity, we do not include such stretched slabs in the statements of our results; this generalization of our results is elementary.

The problem studied here is similar to X-ray tomography on tori, which has been studied by Abouelaz and Rouvière [2,1] and the first author [8], including tensor tomography. However, since we look at geodesics joining boundary points, our set of admissible curves is different.

For tensor tomography results on manifolds, and their applications, we refer to the review [12]. Inverse boundary value problems for PDEs have been considered previously