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Geometric significance of Toeplitz kernels

E. Andruchow [a,d], E. Chiumiento [b,d], G. Larotonda [c,d,*]

[a] *Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina*
[b] *Departamento de Matemática, Facultad de Ciencias Exactas, Universidad de La Plata, Calles 50 y 115, (1900) La Plata, Argentina*
[c] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria (1428) CABA, Argentina*
[d] *Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina*

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Let L^2 be the Lebesgue space of square-integrable functions on the unit circle. We show that the injectivity problem for Toeplitz operators is linked to the existence of geodesics in the Grassmann manifold of L^2. We also investigate this connection in the context of restricted Grassmann manifolds associated to p-Schatten ideals and essentially commuting projections.

* Corresponding author at: Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.

E-mail addresses: eandruch@ungs.edu.ar (E. Andruchow), eduardo@mate.unlp.edu.ar (E. Chiumiento), glaroton@dm.uba.ar (G. Larotonda).

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1. Introduction

Let L^p be the usual Lebesgue spaces of complex-valued functions on the unit circle $\mathbb{T}$. The Grassmann manifold of L^2 is the set of all closed subspaces of L^2. This paper studies the relation between geodesics on the Grassmann manifold of L^2 and the injectivity problem for Toeplitz operators.

To explain this relation, let H^2 be the Hardy space of the unit circle. Recall that the injectivity problem for Toeplitz operators consists in looking for those symbols $\varphi \in L^\infty$ such that the Toeplitz operator T_φ is injective. We relate it to the problem of finding a geodesic on the Grassmann manifold of L^2 which joins two subspaces of the form φH^2 and ψH^2, where φ, ψ are invertible functions in L^∞. More precisely, we will prove that such a geodesic exists if and only if the Toeplitz operator $T_{\varphi\psi^{-1}}$ and its adjoint both have trivial kernel. Furthermore, we will see that these statements are also equivalent to the existence of a minimizing geodesic joining the given subspaces.

The Grassmann manifold of an abstract Hilbert space (i.e. the set consisting of all the closed subspaces) may be identified with the bounded self-adjoint projections. It is an infinite dimensional homogeneous space which can be endowed with a Finsler metric by using the operator norm on each tangent space. Although it is complete with the corresponding rectifiable distance, there are subspaces in the same connected component that cannot be joined by a geodesic (see e.g. [1]). This means that the Hopf–Rinow theorem fails for this manifold. Nevertheless, much information of its geodesics and their minimizing properties are known. The first results date back to the works [21,13,29]; all in the more general framework of self-adjoint projections in C^*-algebras. More recently, there has been progress about the structure of the geodesics in several Grassmann manifolds defined by imposing additional conditions on the subspaces; see for instance [5,3,6] for restricted Grassmann manifolds and [4] for the Lagrangian Grassmann manifold.

In this paper, we turn to a more concrete setting by taking the Hilbert space L^2. This allows us to study the interplay between geodesics, functional spaces and operator theory. In contrast to the invertibility problem for Toeplitz operators, little attention has been paid in the literature to the injectivity problem until recent years. Except for the works of [12,22], the problem remained untreated until the recent works [23–25] (see also the survey [19]). Apart from being an interesting problem in operator theory, in these latter articles there are relevant applications to harmonic analysis, complex analysis and mathematical physics.

The structure of this paper is as follows. In Section 2 we give classical results on Hardy spaces, Toeplitz and Hankel operators to make the article reasonably self-contained. In Section 3 we prove the aforementioned relation between geodesics of the Grassmann manifold of L^2 and the injectivity problem (Theorem 3.4). Then, this result is used to derive an inequality involving the reduced minimum modulus of Toeplitz operators and the norm of a commutator (Theorem 3.8).

In Section 4 we deal with the compact restricted Grassmannian (or Sato Grassmannian). This is a well-known Banach manifold related to KdV equations and loop groups