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Journal of Functional Analysis

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Fractional div-curl quantities and applications to nonlocal geometric equations



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ARTICLE INFO

Article history: Received 6 March 2017 Accepted 26 March 2018 Available online 17 April 2018 Communicated by F. Béthuel

MSC: 42B37 42B30 35R11 58E20 35B65

Keywords: Fractional divergence Fractional div-curl lemma Fractional harmonic maps

ABSTRACT

We investigate a fractional notion of gradient and divergence operator. We generalize the div-curl estimate by Coifman– Lions–Meyer–Semmes to fractional div-curl quantities, obtaining, in particular, a nonlocal version of Wente's lemma.

We demonstrate how these quantities appear naturally in nonlocal geometric equations, which can be used to obtain a theory for fractional harmonic maps analogous to the local theory. Firstly, regarding fractional harmonic maps into spheres, we obtain a conservation law analogous to Shatah's conservation law and give a new regularity proof analogous to Hélein's for harmonic maps into spheres.

Secondly, we prove regularity for solutions to critical systems with nonlocal antisymmetric potentials on the right-hand side. Since the half-harmonic map equation into general target manifolds has this form, as a corollary, we obtain a new proof of the regularity of half-harmonic maps into general target manifolds following closely Rivière's celebrated argument in the local case.

Lastly, the fractional div-curl quantities provide also a new, simpler, proof for Hölder continuity of $W^{s,n/s}$ -harmonic maps into spheres and we extend this to an argument for $W^{s,n/s}$ -harmonic maps into homogeneous targets. This is an analogue of Strzelecki's and Toro–Wang's proof for

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 $\label{eq:https://doi.org/10.1016/j.jfa.2018.03.016} 0022\text{-}1236/ \ensuremath{\odot}\ 2018$ Elsevier Inc. All rights reserved.

n-harmonic maps into spheres and homogeneous target manifolds, respectively.

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Contents

2. Fractional divergence and div-curl lemmas	1.	Introduction	2
3. Fractional div-curl quantities and half-harmonic maps into spheres 8 4. Fractional div-curl quantities and systems with nonlocal antisymmetric potential and half-harmonic maps into general manifolds 12 5. Fractional div-curl quantities and $W^{s,p}$ -harmonic maps into homogeneous manifolds 19 6. Fractional div-curl estimates: proof of Theorem 2.1 and Proposition 2.4 22 Acknowledgments 30 Appendix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2 and Theorem 4.4 30 Appendix B. Euler-Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42	2.	Fractional divergence and div-curl lemmas	5
4. Fractional div-curl quantities and systems with nonlocal antisymmetric potential and half-harmonic maps into general manifolds 12 5. Fractional div-curl quantities and $W^{s,p}$ -harmonic maps into homogeneous manifolds 19 6. Fractional div-curl estimates: proof of Theorem 2.1 and Proposition 2.4 22 Acknowledgments 30 Appendix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2 30 Appendix B. Euler-Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42	3.	Fractional div-curl quantities and half-harmonic maps into spheres	8
harmonic maps into general manifolds 12 5. Fractional div-curl quantities and $W^{s,p}$ -harmonic maps into homogeneous manifolds 19 6. Fractional div-curl estimates: proof of Theorem 2.1 and Proposition 2.4 22 Acknowledgments 30 Appendix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2 30 Appendix B. Euler-Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42	4.	Fractional div-curl quantities and systems with nonlocal antisymmetric potential and half-	
5. Fractional div-curl quantities and W ^{s,p} -harmonic maps into homogeneous manifolds 19 6. Fractional div-curl estimates: proof of Theorem 2.1 and Proposition 2.4 22 Acknowledgments 30 Appendix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2 30 Appendix B. Euler-Lagrange equations for W ^{s,p} -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42		harmonic maps into general manifolds	12
6. Fractional div-curl estimates: proof of Theorem 2.1 and Proposition 2.4 22 Acknowledgments 30 Appendix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2 30 Appendix B. Euler-Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42	5.	Fractional div-curl quantities and $W^{s,p}$ -harmonic maps into homogeneous manifolds \ldots	19
Acknowledgments 30 Appendix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2 30 and Theorem 4.4 30 Appendix B. Euler-Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42	6.	Fractional div-curl estimates: proof of Theorem 2.1 and Proposition 2.4	22
Appendix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2 30 and Theorem 4.4 30 Appendix B. Euler-Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42	Ackno	owledgments	30
and Theorem 4.4 30 Appendix B. Euler-Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42	Apper	ndix A. Nonlocal antisymmetric potential and the optimal gauge: proof of Proposition 4.2	
Appendix B. Euler-Lagrange equations for W ^{s,p} -harmonic maps into homogeneous Riemannian manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel-Lizorkin type space 38 References 42		and Theorem 4.4	30
manifolds: proof of Lemma 5.1 35 Appendix C. An integro-differential Triebel–Lizorkin type space 38 References 42	Apper	ndix B. Euler–Lagrange equations for $W^{s,p}$ -harmonic maps into homogeneous Riemannian	
Appendix C. An integro-differential Triebel–Lizorkin type space 38 References 42		manifolds: proof of Lemma 5.1	35
References	Apper	ndix C. An integro-differential Triebel–Lizorkin type space	38
	Refere	ences	42

1. Introduction

Products of divergence-free and curl-free vector fields, the so-called div-curl-quantities, play a fundamental role in Geometric Analysis. They appear, for example, in the theory of compensated compactness in the form of the div-curl Lemma: let $L^2(\bigwedge^1 \mathbb{R}^n)$ be the L^2 -space of 1-forms on \mathbb{R}^n , or equivalently the space of vector fields $L^2(\mathbb{R}^n, \mathbb{R}^n)$. Given two sequences $\{F_k\}_{k\in\mathbb{N}}, \{G_k\}_{k\in\mathbb{N}}$ in $L^2(\bigwedge^1 \mathbb{R}^n)$ which weakly converge in $L^2(\bigwedge^1 \mathbb{R}^n)$ to Fand G, respectively. In general, there is no reason that the product converges

$$F_k \cdot G_k \xrightarrow{k \to \infty} F \cdot G \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$
 (1.1)

If we know, however, that (in distributional sense) $\operatorname{div}(F_k) = 0$ and $\operatorname{curl}(G_k) = 0$, or more generally assuming compactness of $\operatorname{div}(F_k)$ and $\operatorname{curl}(G_k)$ in H^{-1} , then (1.1) indeed holds true. This phenomenon is known as compensated compactness and its theory was developed by Murat and Tartar in the late seventies [28,29,47–49], see also the more recent [8,10].

In [9] Coifman, Lions, Meyer, and Semmes found a relation between div-curl quantities and the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ (for a definition see Section 6).

Theorem 1.1 (Coifman-Lions-Meyer-Semmes). Let $F \in L^p(\bigwedge^1 \mathbb{R}^n)$ and $g \in \dot{W}^{1,p'}(\mathbb{R}^n)$ where $p \in (1,\infty)$ and $p' = \frac{p}{p-1}$. Then, if

$$\operatorname{div} F = 0,$$

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