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Amenability, Reiter's condition and Liouville property

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A B S T R A C T

We show that the Liouville property and Reiter's condition are equivalent for semigroupoids. This result applies to semigroups as well as semigroup actions. In the special case of measured groupoids and locally compact groupoids, our result proves Kaimanovich's conjecture of the equivalence of amenability and the Liouville property.

1. Introduction

Since the seminal work of von Neumann [38], amenable groups and semigroups have had a profound impact on many areas of mathematics. Amenability of locally compact groups has been shown to be equivalent to many fundamental properties in harmonic

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analysis including the Liouville property which is one of the subjects of the present paper. In operator algebras, amenability plays a pivotal role in their classification (cf. [9,14]) as well as in the recent progress on the Novikov conjecture (see [17,30]). Indeed, amenable groupoids satisfy the Baum–Connes conjecture [37]. Amenability also plays a significant role in the recent development of semigroup C*-algebras relating to some aspects of number theory [10,25,26].

A locally compact group G is amenable if there is a left invariant mean on $L^\infty(G)$. A topological semigroup G is usually called *amenable* if there is a left invariant mean on the algebra $LUC(G)$ of bounded left uniformly continuous functions on G. These two definitions of amenability are equivalent for locally compact groups. The notion of amenability has been extended to group actions by Zimmer [40,41]. For the more general case of groupoids which, among other things, unify both concepts of groups and group actions, it was introduced by Renault [34,1]. Amenable groupoids were defined in terms of Reiter's condition, which stipulates the existence of nets of approximately invariant probability measures and was first formulated by Day [12] for discrete semigroups. For locally compact groups, Reiter's condition is equivalent to amenability and therefore the definition of an amenable groupoid is a natural extension of the group case. However, for topological semigroups, the question of whether Reiter's condition follows from amenability as defined previously appears to be open [23, p.321].

The equivalence of amenability and the Liouville property for σ-compact locally compact groups was first conjectured by Furstenberg [16] and proved by Rosenblatt [36], Kaimanovich and Vershik [21]. More recently, Kaimanovich introduced the Liouville property for groupoids in [20] and conjectured its equivalence to amenability, having proved that the former implies the latter. For semigroups, the Liouville property for abelian semigroups has been studied in [11,24,32], but its connection to amenability has not been the subject of investigation before.

Our main objective in this paper is to clarify the relationships of amenability, Reiter's condition and the Liouville property in the setting of semigroupoids, which subsumes and provides a unified treatment to the important cases of groupoids, semigroups and transformation semigroups. We introduce and study the Liouville property and Reiter's condition for semigroupoids. We prove that a semigroupoid possesses the Liouville property if and only if it satisfies Reiter's condition (Theorems 4.1, 4.2, 5.3, 5.6). An immediate consequence is the equivalence of the Liouville property and Reiter's condition for semigroup actions (Theorem 6.7) as well as the equivalence of amenability and the Liouville property for discrete semigroups (Theorem 6.10) and also, for both measured groupoids and topological groupoids (Theorems 6.1, 6.3), the latter proves a conjecture of Kaimanovich in [20]. We thank Vadim Kaimanovich for informing us, after we have written this paper, that his conjecture for *measured groupoids* has also been proved by Theo Bühler and himself in an unpublished note. Our result includes the case of topological groupoids, which requires some refinements of Reiter's condition.

A Riemannian manifold is said to have the Liouville property if it does not admit non-constant bounded harmonic functions. Examples include complete manifolds with