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Sharp gradient estimates for quasilinear elliptic equations with $p(x)$ growth on nonsmooth domains

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In this paper, we study quasilinear elliptic equations with the nonlinearity modelled after the $p(x)$-Laplacian on nonsmooth domains and obtain sharp Calderón–Zygmund type estimates in the variable exponent setting. In a recent work of [12], the estimates obtained were strictly above the natural exponent and hence there was a gap between the natural energy estimates and estimates above $p(x)$, see (1.3) and (1.4). Here, we bridge this gap to obtain the end point case of the estimates obtained in [12], see (1.5). In order to do this, we have to obtain significantly improved a priori estimates below $p(x)$, which is the main contribution of this paper. We also improve upon the previous results by obtaining the estimates for a larger class of domains than what was considered in the literature.

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1. Introduction

Calderón–Zygmund theory was first developed for the Poisson equation in [16], which related the integrability of the gradient of the solution for the Poisson equation with that of the associated data. This represented the starting point of obtaining a priori estimates in Sobolev spaces for elliptic and parabolic equations.

All the estimates mentioned in this introduction are quantitative in nature, but to avoid being too technical, we only recall the qualitative nature of the bounds. This is sufficient to highlight the nature of the results that we will prove in this paper.

For the problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

T. Iwaniec in [31] established the Calderón–Zygmund type estimates, in particular he proved the following a priori relation

$$|\mathbf{f}| \in L^q_{loc} \Longrightarrow |\nabla u| \in L^q_{loc} \qquad \text{for all } q > p.$$

After this pioneering work, there have been numerous publications which extended these estimates to other quasilinear elliptic and parabolic equations with the constant p-growth, see [3,9,15,20,23,36,42] and references therein. In this paper, we are interested in obtaining Calderón–Zygmund type bounds for the problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div}(|\mathbf{f}|^{p(x)-2}\mathbf{f}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, and the quasilinear operator $\mathcal{A}(x, \nabla u)$ is modelled after well known $p(x)$-Laplacian operator having the form $|\nabla u|^{p(x)-2}\nabla u$. See Section 2 for the precise assumptions on $\mathcal{A}(\cdot,\cdot)$, $p(\cdot)$ and Ω. For more on the importance of variable exponent problems, see [43,44,47,17,7,30] and the references therein.

The first estimate for the $p(x)$-Laplacian was obtained by Acerbi and Mingione in [2], wherein they obtained a local Calderón–Zygmund type estimate by proving

$$|\mathbf{f}|^{p(\cdot)} \in L^q_{loc} \Longrightarrow |\nabla u|^{p(\cdot)} \in L^q_{loc} \qquad \text{for all } q \in (1, \infty) \tag{1.2}$$

under the assumption that the variable exponent $p(\cdot)$ satisfies $\lim_{r\to 0} \rho(r) \log\left(\frac{1}{r}\right) = 0$ (see Section 2.1 for the relation between ρ and $p(\cdot)$). This work was subsequently extended in [8] to parabolic systems.

This estimate was further improved upon in [13] for more general equations of the form (1.1), to obtain global Calderón–Zygmund type estimates, provided the nonlinearity $\mathcal{A}(x, \zeta)$ satisfied a small BMO (bounded mean oscillation) condition with respect to x

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