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Stable solutions of symmetric systems involving hypoelliptic operators

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ABSTRACT

Let X and Y be two noncommuting vector fields in an open set Ω in a manifold $\mathbb{M}$ equipped with a sub-Riemannian structure. We examine stable solutions of the following symmetric system

$$\Delta_{XY} u_i = H_i(u_1, \cdots, u_m) \quad \text{in} \quad \Omega \quad \text{for} \quad 1 \le i \le m,$$

when the operator Δ_{XY} is the Hörmander's operator given by $\Delta_{XY}(\cdot) := X(X\cdot) + Y(Y\cdot)$ and $H_i \in C^1(\mathbb{R}^m)$. We prove the following identity for any $w \in C^2(\Omega)$

$$|\nabla_{XY} Xw|^2 + |\nabla_{XY} Yw|^2 - |X|\nabla_{XY} w||^2 - |Y|\nabla_{XY} w||^2$$
$$= \begin{cases} |\nabla_{XY} w|^2 \left[\mathcal{A}^2 + \mathcal{B}^2\right] & \text{in } \{|\nabla_{XY} w| > 0\} \cap \Omega, \\ 0 \ a.e. & \text{in } \{|\nabla_{XY} w| = 0\} \cap \Omega, \end{cases}$$

where $\mathcal{A}$ is the intrinsic curvature of the level sets of w and $\mathcal{B}$ is connected with the intrinsic normal and the intrinsic tangent direction to the level sets and also with the Lie bracket $[X, Y]$. We then apply this to establish a geometric Poincaré inequality for stable solutions of the above system for general vector fields X and Y. This inequality enables us to analyze the level sets of stable solutions. In addition, we provide certain reduction of dimensions results which can be regarded as counterparts of the classical De Giorgi type results. This is remarkable since the classical one-dimensional symmetry

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results do not hold for general vector fields. Our approaches can be applied, but not limited, to the Grushin vector fields $X = (1,0)$ and $Y = (0,x)$ in $\mathbb{R}^2$ and the Heisenberg vector fields $X = (1,0,-\frac{y}{2})$ and $Y = (0,1,\frac{x}{2})$ in $\mathbb{R}^3$ and their multidimensional extensions. These specific vector fields generate nonelliptic operators which are hypoelliptic.

1. Introduction

It is by now standard that any twice continuous differentiable solution of the Laplace's equation is an analytic function. An analogous property applies for any elliptic equations and systems with analytic coefficients. Inspired by this fact, the question of describing more general differential operators $P(x,D)$ having the property that any solution of the linear equation

$$P(x,D)u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^n \text{ (or a manifold)}, \tag{1.1}$$

is in $C^\infty(\Omega)$ when $f \in C^\infty(\Omega)$ was introduced by Schwartz [51] and Hörmander [39–41] and it has been studied extensively in the literature since then. The literature in this context is too vast to give more than a few representative references [20,31,32,36,42,43,48,49]. Differential operators satisfying this property are called hypoelliptic. In addition, the heat equation operator

$$P(x,D)u = u_t - c\Delta_x u, \tag{1.2}$$

when $c > 0$ is hypoelliptic but not elliptic and the wave equation operator

$$P(x,D)u = u_{tt} - c^2\Delta_x u, \tag{1.3}$$

when $c \neq 0$ is not hypoelliptic. This implies that hypoellipticity is not necessarily ellipticity. However, every elliptic operator with C^∞ coefficients is hypoelliptic. Generally speaking, when the coefficients in the above operator P are constant, a complete algebraic characterization of hypoelliptic operators is derived by Hörmander in 1950s, see [39]. However, the case of nonconstant coefficients is more challenging and some sufficient and necessary conditions are given by various authors, see [40–43,49] and references therein.

In 1970, Grushin [36] studied the following class of differential operators with polynomial coefficients which are not elliptic, but satisfy the Hörmander's conditions for hypoellipticity under certain assumptions;

$$P(x,D)u = \sum_{|\alpha|+|\beta|\leq N, |\gamma|\leq N\delta} a_{\alpha\beta\gamma}(x')^\gamma D^\alpha_{x'} D^\beta_{x''} u, \tag{1.4}$$

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