Journal of Functional Analysis 274 (2018) 3033–3055

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

On diagonal actions of branch groups and the corresponding characters

Artem Dudko [a,*], Rostislav Grigorchuk [b]

[a] *IMPAN, Warsaw, Poland*
[b] *Texas A&M University, College Station, TX, USA*

ARTICLE INFO

Article history:
Received 27 December 2014
Accepted 24 February 2018
Available online 6 March 2018
Communicated by Stefaan Vaes

MSC:
20C15
20E08
37A05

Keywords:
Non-free action
Character
Factor representation
Branch group

ABSTRACT

We introduce notions of absolutely non-free and perfectly non-free group actions and use them to study the associated unitary representations. We show that every weakly branch group acting on a regular rooted tree acts absolutely non-freely on the boundary of the tree. Using this result and the symmetrized diagonal actions we construct for every countable branch group infinitely many different ergodic perfectly non-free actions, infinitely many II_1-factor representations, and infinitely many continuous ergodic invariant random subgroups.

1. Introduction

Branch groups were introduced by the second author in [12]. The class of branch groups contains many examples of groups with remarkable properties. Among them is the group of intermediate growth constructed in [10] and studied in [11] to answer Mil-

* Corresponding author.
E-mail addresses: adudko@impan.pl (A. Dudko), grigorch@math.tamu.edu (R. Grigorchuk).

https://doi.org/10.1016/j.jfa.2018.02.016
0022-1236/

nor's question on growth and Day's question on existence of a non-elementary amenable group. Weakly branch groups form a more general class of groups acting on spherically homogeneous rooted trees. It contains, for instance, the Basilica group, which is the first example of amenable but not subexponentially amenable group and is an iterated monodromy group of the map $f(z) = z^2 - 1$ [19]. Branch and weakly branch groups have many important applications in group theory, combinatorics, spectral theory, holomorphic dynamics, probability theory, *etc.* This paper is dedicated to studying diagonal actions of weakly branch groups and associated characters and factor representations. Some of our results hold for all branch and weakly branch groups, however, to simplify the presentation we state and prove the results for branch and weakly branch groups acting on regular rooted trees.

Our results are related to the notion of non-free actions studied by A. Vershik in [26] and [27]. An action of a group G on a Lebesgue space (also called standard probability space) is *totally non-free* (TNF) if the sigma-algebra generated by the sets $\mathrm{Fix}(g)$ of fixed points of elements $g \in G$ is the whole sigma-algebra Σ of measurable subsets of this space. An action of G is called *extremely non-free* (ENF) if on a set of a full measure, different points have different stabilizers. In [26], Theorem 8, Vershik showed that for countable groups these two notions are equivalent. It is known [13] that the action of a weakly branch group G on the boundary ∂T of the associated rooted tree T equipped with the unique invariant measure μ is totally non-free. Here we show that it has a stronger non-freeness. Namely, the sets $\mathrm{Fix}(g), g \in G$ approximate measurable subsets of ∂T arbitrarily well in μ-measure. We call actions satisfying this property *absolutely non-free* (ANF). We also introduce the notion of *perfect non-freeness* (PNF) of actions, which is stronger than total non-freeness, but weaker than absolute non-freeness. The relation between the different notions of non-freeness for countable groups is summarized in the following diagram:

$$\mathrm{ANF} \;\Rightarrow\; \mathrm{PNF} \;\Rightarrow\; \mathrm{TNF} \;\Leftrightarrow\; \mathrm{ENF}. \tag{1}$$

As a corollary of our results, we obtain that an appropriate factor of the diagonal action of a countable weakly branch group G on $(\partial T, \mu)^{\otimes n}$ for any $n \in \mathbb{N}$ is perfectly non-free. Another consequence is that every branch group has infinitely many pairwise non-isomorphic ergodic perfectly (and hence totally) non-free measure-preserving actions, and therefore infinitely many ergodic continuous invariant random subgroups. We apply the notion of a perfectly non-free action to study associated characters and factor representations.

By a character χ on a group G we mean a normalized (by $\chi(e) = 1$ for the unit e of the group), positive semi-definite complex valued function on G that is constant on conjugacy classes. Indecomposable characters are extreme points in the simplex of characters and are in one-to-one correspondence with quasi-equivalence classes of finite type factor representations of G. Group characters and factor representations is a classical topic of the theory of unitary representations of locally compact groups and operator