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Poincaré, modified logarithmic Sobolev and isoperimetric inequalities for Markov chains with non-negative Ricci curvature

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ABSTRACT

We study functional inequalities for Markov chains on discrete spaces with entropic Ricci curvature bounded from below. Our main results are that when curvature is non-negative, but not necessarily positive, the spectral gap, the Cheeger isoperimetric constant and the modified logarithmic Sobolev constant of the chain can be bounded from below by a constant that only depends on the diameter of the space, with respect to a suitable metric. These estimates are discrete analogues of classical results of Riemannian geometry obtained by Li and Yau, Buser and Wang.

1. Introduction

Ricci curvature bounds play an important role in geometric analysis on Riemannian manifolds. For instance, a lower bound on the curvature by a strictly positive constant

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entails many interesting properties for the manifold, most notably Harnack inequalities, bounds on the eigenvalues of the Laplacian, concentration bounds and isoperimetric inequalities.

In light of this wide range of implications, considerable effort has been put into developing a notion of Ricci curvature lower bounds for non-smooth spaces. Bakry and Émery [1] proposed a curvature condition for general Markov diffusion operators via the so-called Γ-calculus. Lott–Villani [27] and Sturm [38] presented an approach that applies to (geodesic) metric measure spaces. Such a space has Ricci curvature bounded below by a constant κ provided the entropy is κ-convex along geodesics in the Wasserstein space of probability measures. Subsequently, many of the classical results relating curvature bounds to functional inequalities have been generalized to such 'continuous' non-smooth spaces, we refer to [2,39] for an overview.

In recent years, there has been a strong interest in developing an analogous theory for discrete spaces. Unfortunately, the Lott–Sturm–Villani theory does not apply, and a number of alternative notions of Ricci bounds have been proposed, see for instance [7,15,34]. In this work, we will focus on the notion of *entropic Ricci curvature bounds* put forward in [28,12] that applies to finite Markov chains and seems to be particularly well suited to study discrete functional inequalities. Here the key point is to replace the L^2-Wasserstein distance with a new transportation distance $\mathcal{W}$ in the definition of Lott–Sturm–Villani. It has been shown in [12] that a strictly positive entropic Ricci curvature lower bound implies a spectral gap estimate, a modified logarithmic Sobolev inequality and an analogue of Talagrand's transport cost inequality.

In the present work, we are interested in the situation where the curvature is bounded from below but not strictly positive. We show that in this situation relatively weak extra information (for instance a bound on the diameter of the space) still allows one to establish strong functional inequalities.

To state our main results we consider an irreducible and reversible continuous time Markov chain on a finite space $\mathcal{X}$ whose generator is given by

$$L\psi(x) = \sum_{y\in\mathcal{X}} \big(\psi(y) - \psi(x)\big)Q(x,y)\ ,$$

where $Q(x,y)$ are the transition rates between x and y and let π be the unique reversible probability measure.

For the purpose of this introduction we state our main results for simplicity under the assumption that the chain has non-negative entropic Ricci curvature. We shall actually derive more general statements allowing for a negative curvature bound in the main text. We refer to Section 2 for a precise definition of entropic Ricci curvature bounds and the functional inequalities we consider.

The first result establishes an isoperimetric inequality using information on the spectral gap (see Theorem 4.1 below).