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The multidimensional truncated moment problem: Atoms, determinacy, and core variety

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ABSTRACT

This paper is about the moment problem on a finite-dimensional vector space of continuous functions. We investigate the structure of the convex cone of moment functionals (supporting hyperplanes, exposed faces, inner points) and treat various important special topics on moment functionals (determinacy, set of atoms of representing measures, core variety).

1. Introduction

Let N be a finite subset of $\mathbb{N}_0^n$, $n \in \mathbb{N}$, and $\mathsf{A} = \{x^\alpha : \alpha \in \mathsf{N}\}$, $\mathcal{A} = \operatorname{Lin} \mathsf{A}$ the span of associated monomials, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Suppose that

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$\mathcal{K}$ is a closed subset of $\mathbb{R}^n$. Let $s = (s_\alpha)_{\alpha \in \mathsf{N}}$ be a real sequence and let L_s denote the corresponding Riesz functional on $\mathcal{A}$ defined by $L_s(x^\alpha) := s_\alpha$, $\alpha \in \mathsf{N}$.

The truncated moment problem asks: *When does there exist a (positive) Radon measure μ on $\mathcal{K}$ such that x^α is μ-integrable and*

$$s_\alpha = \int_{\mathbb{R}^n} x^\alpha \, d\mu \quad \text{for all} \quad \alpha \in \mathsf{N}? \tag{1}$$

Clearly, (1) is equivalent to

$$L_s(f) = \int_{\mathcal{K}} f(x) \, d\mu \quad \text{for} \quad f \in \mathcal{A}. \tag{2}$$

The Richter–Tchakaloff theorem (Proposition 4) implies that in the affirmative case there is always a finitely atomic measure μ satisfying (1) and (2).

The multidimensional truncated moment problem was first studied in the unpublished Thesis of J. Matzke [12] and by R. Curto and L. Fialkow [4], [5], see [11] for a nice survey. The one-dimensional case is treated in the monographs [8], [10], see also [1] and [17].

In the present paper we consider the truncated moment problem in a more general setting. That is, we study moment functionals on a finite-dimensional vector space E of real-valued continuous functions on a locally compact topological Hausdorff space $\mathcal{X}$. This covers a large number of interesting examples. For instance, we may take for $\mathcal{X}$ a closed subset of $\mathbb{R}^n$ and for E the linear span of finitely many exponentials or rational functions or eigenfunctions of some differential operator. The truncated $\mathcal{K}$-moment problem for polynomials is obtained by letting E the vector space of restrictions $f\lceil\mathcal{K}$ of functions $f \in \mathcal{A}$ to $\mathcal{X} := \mathcal{K}$. As discussed in Remark 2 below, in this case a sequence is a truncated $\mathcal{K}$-moment sequence as defined above if and only if it is a moment sequence in the sense of Definition 1 below. One may specialize further by setting $\mathcal{K} = \mathbb{R}^n$ and $\mathsf{N} = \{\alpha \in \mathbb{N}_0^n : \alpha_1 + \cdots + \alpha_n \leq 2d\}$. Then $\mathcal{A}$ is the vector space $\mathbb{R}[x_1, \ldots, x_n]_{2d}$ of real polynomials in n variables of degree at most $2d$. This is our guiding example. All theorems proved in this paper give new results even in this case.

Let us briefly describe the structure and the contents of this paper. In Section 2 we recall basic notation, definitions and facts on moment sequences and moment functionals. Let L be a moment functional on E. The set $\mathcal{W}(L)$ of possible atoms of representing measures of L is investigated in Section 3. In Section 4, we characterize the determinacy of L in terms of the set $\mathcal{W}(L)$ (Theorem 16). Three other important notions associated with L are studied in Sections 5 and 6. These are the cone $\mathcal{N}_+(L)$ of non-negative functions of E which are annihilated by L, the zero set $\mathcal{V}_+(L)$ of $\mathcal{N}_+(L)$ and the core variety $\mathcal{V}(L)$ introduced by L. Fialkow [6]. It is easily seen that $\mathcal{W}(L) \subseteq \mathcal{V}_+(L)$. Example 13 shows that both sets are not equal in general.

We prove that equality holds if and only if the moment sequence of L lies in the relative interior of an exposed face of the moment cone (Theorem 30). One of our main results