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On weaker notions of nonlinear embeddings between Banach spaces

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In this paper, we study nonlinear embeddings between Banach spaces. More specifically, the goal of this paper is to study weaker versions of coarse and uniform embeddability, and to provide suggestive evidences that those weaker embeddings may be stronger than one would think. We do such by proving that many known results regarding coarse and uniform embeddability remain valid for those weaker notions of embeddability.

1. Introduction

The study of Banach spaces as metric spaces has recently increased significantly, and much has been done regarding the uniform and coarse theory of Banach spaces in the past two decades. In particular, the study of coarse and uniform embeddings has been receiving a considerable amount of attention (e.g., [4], [5], [9], [12], [15], [18]). These notes are dedicated to the study of several different notions of nonlinear embeddings between

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Banach spaces, and our main goal is to provide the reader with evidences that those kinds of embeddings may not be as different as one would think.

Let (M, d) and (N, ∂) be metric spaces, and consider a map $f : (M, d) \to (N, \partial)$. For each $t \geq 0$, we define the *expansion modulus of* f as

$$\omega_f(t) = \sup\{\partial(f(x), f(y)) \mid d(x, y) \leq t\},$$

and the *compression modulus of* f as

$$\rho_f(t) = \inf\{\partial(f(x), f(y)) \mid d(x, y) \geq t\}.$$

Hence, $\rho_f(d(x, y)) \leq \partial(f(x), f(y)) \leq \omega_f(d(x, y))$, for all $x, y \in M$. The map f is uniformly continuous if and only if $\lim_{t\to 0^+} \omega_f(t) = 0$, and f^{-1} exists and it is uniformly continuous if and only if $\rho_f(t) > 0$, for all $t > 0$. If both f and its inverse f^{-1} are uniformly continuous, f is called a *uniform embedding*. The map f is called *coarse* if $\omega_f(t) < \infty$, for all $t \geq 0$, and *expanding* if $\lim_{t\to\infty} \rho_f(t) = \infty$. If f is both expanding and coarse, f is called a *coarse embedding*. A map which is both a uniform and a coarse embedding is called a *strong embedding*.

Those notions of embeddings are fundamentally very different, as coarse embeddings deal with the large scale geometry of the metric spaces concerned, and uniform embeddings only deal with their local (uniform) structure. Although those notions are fundamentally different, it is still not known whether the existence of those embeddings are equivalent in the Banach space setting. Precisely, the following problem remains open.

Problem 1.1. Let X and Y be Banach spaces. Are the following equivalent?

(i) X coarsely embeds into Y.
(ii) X uniformly embeds into Y.
(iii) X strongly embeds into Y.

It is known that Problem 1.1 has a positive answer if Y is either ℓ_∞ (see [11], Theorem 5.3) or ℓ_p, for $p \in [1, 2]$ (see [18], Theorem 5, and [20], page 1315). C. Rosendal made some improvements on this problem by showing that if X uniformly embeds into ℓ_p, then X strongly embeds into ℓ_p, for all $p \in [1, \infty)$. This result can be generalized by replacing ℓ_p with any minimal Banach space (see [4], Theorem 1.2(i)).

Natural weakenings for the concepts of coarse and uniform embeddings were introduced in [22], and, as it turns out, those weaker notions are rich enough for many applications. Given a map $f : (M, d) \to (N, \partial)$ between metric spaces, we say that f is *uncollapsed* if there exists some $t > 0$ such that $\rho_f(t) > 0$. The map f is called *solvent* if, for each $n \in \mathbb{N}$, there exists $R > 0$, such that

$$d(x, y) \in [R, R + n] \quad \text{implies} \quad \partial(f(x), f(y)) > n,$$

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