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The space of stable weak equivalence classes of measure-preserving actions

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A R T I C L E I N F O

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The concept of (stable) weak containment for measure-preserving actions of a countable group Γ is analogous to the classical notion of (stable) weak containment of unitary representations. If Γ is amenable then the Rokhlin lemma shows that all essentially free actions are weakly equivalent. However if Γ is non-amenable then there can be many different weak and stable weak equivalence classes. Our main result is that the set of stable weak equivalence classes naturally admits the structure of a Choquet simplex. For example, when $\Gamma = \mathbb{Z}$ this simplex has only a countable set of extreme points but when Γ is a nonamenable free group, this simplex is the Poulsen simplex. We also show that when Γ contains a nonabelian free group, this simplex has uncountably many strongly ergodic essentially free extreme points.

Contents

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1. Introduction

A. Kechris introduced the notion of weak containment for group actions as an analogue of weak containment for unitary representations [21, II.10 (C)]. Given a countable group Γ and probability measure-preserving (pmp) actions $\mathbf{a} := \Gamma \curvearrowright^a (X, \mu)$, $\mathbf{b} := \Gamma \curvearrowright^b (Y, \nu)$ on standard probability spaces, we say $\mathbf{a}$ is **weakly contained** in $\mathbf{b}$ (denoted $\mathbf{a} \prec \mathbf{b}$) if for every finite measurable partition $\{P_i\}_{i=1}^n$ of X, finite $S \subseteq \Gamma$ and $\epsilon > 0$ there exists a measurable partition $\{Q_i\}_{i=1}^n$ of Y satisfying

$$|\mu(\gamma^a P_i \cap P_j) - \nu(\gamma^b Q_i \cap Q_j)| < \epsilon$$

for all $\gamma \in S$ and $1 \leq i, j \leq n$ (where the action of $\Gamma \curvearrowright^a X$ is denoted $\gamma^a x$ for $\gamma \in \Gamma, x \in X$ for example). We say $\mathbf{a}$ is **weakly equivalent** to $\mathbf{b}$, denoted $\mathbf{a} \sim \mathbf{b}$, if both $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{b} \prec \mathbf{a}$.

The Rokhlin Lemma is essentially equivalent to the statement that for the group $\Gamma = \mathbb{Z}$ all essentially free[3] pmp actions are weakly equivalent. Indeed, as remarked in [22], this statement holds for all countable amenable groups. However it fails for nonamenable groups because strong ergodicity is an invariant of weak equivalence [21, Prop. 10.6]. This motivates the problem of providing a description of the set of all weak equivalence classes, denoted by $\mathcal{W}_\Gamma$, for a given group Γ.

We start with an equivalent definition of weak containment. Let **Cantor** denote any space homeomorphic to a Cantor set. Let Γ act on $\mathbf{Cantor}^\Gamma$ by $(\gamma x)(f) = x(\gamma^{-1} f)$. Let $\mathrm{Prob}_\Gamma(\mathbf{Cantor}^\Gamma)$ denote the space of all Γ-invariant Borel probability measures on $\mathbf{Cantor}^\Gamma$ equipped with the weak* topology. It is well-known that $\mathrm{Prob}_\Gamma(\mathbf{Cantor}^\Gamma)$ is a

[3] An action is essentially free if almost every point has trivial stabilizer.