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MINIMAL AND MAXIMAL MATRIX CONVEX SETS

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Abstract. To every convex body $K \subseteq \mathbb{R}^d$, one may associate a minimal matrix convex set $\mathcal{W}^{\min}(K)$, and a maximal matrix convex set $\mathcal{W}^{\max}(K)$, which have K as their ground level. The main question treated in this paper is: under what conditions on a given pair of convex bodies $K, L \subseteq \mathbb{R}^d$ does $\mathcal{W}^{\max}(K) \subseteq \mathcal{W}^{\min}(L)$ hold? For a convex body K, we aim to find the optimal constant $\theta(K)$ such that $\mathcal{W}^{\max}(K) \subseteq \theta(K) \cdot \mathcal{W}^{\min}(K)$; we achieve this goal for all the ℓ^p unit balls, as well as for other sets. For example, if $\overline{\mathbb{B}}_{p,d}$ is the closed unit ball in $\mathbb{R}^d$ with the ℓ^p norm, then

$$\theta(\overline{\mathbb{B}}_{p,d}) = d^{1-|1/p-1/2|}.$$

This constant is sharp, and it is new for all $p \neq 2$. Moreover, for some sets K we find a minimal set L for which $\mathcal{W}^{\max}(K) \subseteq \mathcal{W}^{\min}(L)$. In particular, we obtain that a convex body K satisfies $\mathcal{W}^{\max}(K) = \mathcal{W}^{\min}(K)$ if and only if K is a simplex.

These problems relate to dilation theory, convex geometry, operator systems, and completely positive maps. We discuss and exploit these connections as well. For example, our results show that every d-tuple of self-adjoint operators of norm less than or equal to 1, can be dilated to a commuting family of self-adjoints, each of norm at most $\sqrt{d}$. We also introduce new explicit constructions of these (and other) dilations.

1. Introduction

1.1. **Overview.** This paper treats containment problems for matrix convex sets. A *matrix convex set* in d-variables is a set $\mathcal{S} = \cup_n \mathcal{S}_n$, where every $\mathcal{S}_n$ consists of d-tuples of $n \times n$ matrices, that is closed under direct sums, unitary conjugation, and the application of completely positive maps. Matrix convex sets are closely connected to operator systems and have been investigated for several decades. Recently, they appeared in connection with the interpolation problem for UCP maps (see, e.g., [6, 11, 24]), and also in the setting of relaxation of spectrahedral containment problems [11, 13].

Given a closed convex set $K \subseteq \mathbb{R}^d$, one can define several matrix convex sets $\mathcal{S}$ such that $\mathcal{S}_1 = K$. One may ask, to what extent does the "ground level" $\mathcal{S}_1$ determine the structure and the size of $\mathcal{S}$? For example: given two matrix convex sets $\mathcal{S} = \cup_n \mathcal{S}_n$ and $\mathcal{T} = \cup_n \mathcal{T}_n$, what does containment at the first level $\mathcal{S}_1 \subseteq \mathcal{T}_1$ imply about the relationship between $\mathcal{S}$ and $\mathcal{T}$? Of course, there is (usually) no reason that $\mathcal{S} \subseteq \mathcal{T}$ would follow as a consequence of $\mathcal{S}_1 \subseteq \mathcal{T}_1$, but in many cases — given some conditions on $\mathcal{S}_1$ — one can find a constant C such that

$$\mathcal{S}_1 \subseteq \mathcal{T}_1 \Longrightarrow \mathcal{S} \subseteq C \cdot \mathcal{T}.$$

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