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Hamilton differential Harnack inequality and W-entropy for Witten Laplacian on Riemannian manifolds

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A B S T R A C T

In this paper, we prove the Hamilton differential Harnack inequality for positive solutions to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition, where $m \in [n, \infty)$ and $K \geq 0$ are two constants. Moreover, we introduce the W-entropy and prove the W-entropy formula for the fundamental solution of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition and on compact manifolds equipped with $(-K, m)$-super Ricci flows.

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1. Introduction

Differential Harnack inequality is an important tool in the study of geometric PDEs. Let M be an n dimensional complete Riemannian manifold, u a positive solution to the heat equation

$$\partial_t u = \Delta u. \tag{1}$$

In their famous paper [6], Li and Yau proved that if $Ric \geq -K$, where $K \geq 0$ is a positive constant, then for all $\alpha > 1$,

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{\sqrt{2}(\alpha - 1)}. \tag{2}$$

In particular, if $Ric \geq 0$, then taking $\alpha \to 1$, the Li–Yau differential Harnack inequality holds

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}. \tag{3}$$

In [4], Hamilton proved a dimension free differential Harnack inequality on compact Riemannian manifolds with Ricci curvature bounded from below. More precisely, if M is a compact Riemannian manifold with

$$Ric \geq -K,$$

then, for any positive and bounded solution u to the heat equation (1), it holds

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \log(A/u), \tag{4}$$

where $A := \sup\{u(t, x) : x \in M, t \geq 0\}$. Indeed, the same result holds on complete Riemannian manifolds with Ricci curvature bounded from below. Under the same condition $Ric \geq -K$, Hamilton also proved the following differential Harnack inequality for any positive solution to the heat equation (1)

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{n}{2t} e^{4Kt}. \tag{5}$$

In particular, when $K = 0$, the above inequality reduces to the Li–Yau Harnack inequality (3) on complete Riemannian manifolds with non-negative Ricci curvature. Moreover, Hamilton [4] proved that, on compact Riemannian manifolds with $Ric \geq -K$, then any positive and bounded solution of the heat equation $\partial_t u = \Delta u$ with $0 < u \leq A$ satisfies

$$\frac{\partial_t u}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1 - e^{-Kt}} \left[n + 4 \log(A/u)\right]. \tag{6}$$