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Trace theorems for functions of bounded variation in metric spaces

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ABSTRACT

In this paper we show existence of traces of functions of bounded variation on the boundary of a certain class of domains in metric measure spaces equipped with a doubling measure supporting a 1-Poincaré inequality, and obtain L^1 estimates of the trace functions. In contrast with the treatment of traces given in other papers on this subject, the traces we consider do not require knowledge of the function in the exterior of the domain. We also establish a Maz'ya-type inequality for functions of bounded variation that vanish on a set of positive capacity.

1. Introduction

The Dirichlet problem for functions of least gradient on a domain Ω is to find a function u that minimizes the energy $\int_\Omega |\nabla u|$, or the BV energy obtained from this by

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relaxation, amongst the class of all functions of bounded variation (BV functions) with prescribed boundary values on $\partial\Omega$. In order to make sense of the problem, one needs to know how to extend the function u (which is a priori defined only on Ω) to $\partial\Omega$, and one needs to know for which boundary data the problem is solvable. The focus of the current paper is to study the issue of how a function in the class $\mathrm{BV}(\Omega)$ can be extended to $\partial\Omega$, that is, whether it has a *trace* on $\partial\Omega$. The second question needs very strong geometric conditions on Ω even in the Euclidean setting, where minimizing the BV energy is equivalent to the Dirichlet problem for the so-called 1-Laplacian. In the Euclidean setting the situation is reasonably well understood from the works [7,15,43], but even in the more general weighted Euclidean setting the situation is far from clear. The works [7,15] required the domains to be Euclidean with Lipschitz boundary, and this restriction does not present itself as being natural in the more general weighted Euclidean spaces where the weights are in the Muckenhoupt $\mathcal{A}_1$-class but not smooth. The work [43], based on the notion of positive mean curvature of the boundary, is tied up with the flatness of the ambient Euclidean space, mere smooth Riemannian structure being insufficient.

Nonetheless, the approach of [43] does give an indication of how one might proceed in the weighted Euclidean case, and more generally in other Riemannian manifold settings where the manifold is not flat, and indeed in more general metric measure spaces. However, we first need a solid understanding of the notion of trace of the class of BV functions on a given domain. This functional analytic groundwork is developed in the current paper, and in the paper [35], which uses the results and tools developed in this paper. The authors of the present paper, in collaboration with others, are currently using the tools developed here and in [35] to study a notion of positively curved boundary of a domain that is an analog of [43] in the non-flat setting, see [33].

In the classical Euclidean setting, a standard result is that if the boundary $\partial\Omega$ can be presented locally as a Lipschitz graph, then the trace of a BV function exists. Classical treatments of boundary traces of BV functions can be found in e.g. [3, Chapter 3], [15, Chapter 2], and [7], and similar results for Carnot groups are given in [44]. The results in Lipschitz domains are obtained by first locally “straightening” the boundary of the domain, and then applying trace results regarding BV functions on the upper half-space to obtain trace results in the Lipschitz setting. We seek to extend the concept of traces of BV functions beyond such Lipschitz domains, and beyond the Euclidean and Carnot group setting to the non-smooth metric setting.

In the setting of general metric measure spaces, where the standard assumptions are a doubling measure and a Poincaré inequality, traces of BV functions have only been studied recently. In [19], results on traces are obtained by making rather strong assumptions on the extendability of BV functions from the domain to the whole space. These extendability properties are satisfied by *uniform* domains, which are a natural generalization of Euclidean domains with Lipschitz boundaries. In [32], results on traces are obtained in more general domains but with stronger assumptions on the metric space. In these papers, the existence of traces of a function in $\mathrm{BV}(\Omega)$ is obtained from