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Lower bounds for the first eigenvalue of the magnetic Laplacian

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ABSTRACT

We consider a Riemannian cylinder Ω endowed with a closed potential 1-form A and study the magnetic Laplacian Δ_A with magnetic Neumann boundary conditions associated with those data. We establish a sharp lower bound for the first eigenvalue and show that the equality characterizes the situation where the metric is a product. We then look at the case of a planar domain bounded by two closed curves and obtain an explicit lower bound in terms of the geometry of the domain. We finally discuss sharpness of this last estimate.

1. Introduction

Let (Ω, g) be a compact Riemannian manifold with boundary. Consider the trivial complex line bundle $\Omega \times \mathbf{C}$ over Ω; its space of sections can be identified with $C^\infty(\Omega, \mathbf{C})$, the space of smooth complex valued functions on Ω. Given a smooth real 1-form A on Ω we define a connection ∇^A on $C^\infty(\Omega, \mathbf{C})$ as follows:

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$$\nabla_X^A u = \nabla_X u - iA(X)u \tag{1}$$

for all vector fields X on Ω and for all $u \in C^\infty(\Omega, \mathbf{C})$; here ∇ is the Levi-Civita connection associated to the metric g of Ω. The operator

$$\Delta_A = (\nabla^A)^\star \nabla^A \tag{2}$$

is called the *magnetic Laplacian* associated to the magnetic potential A, and the smooth two form

$$B = dA$$

is the associated *magnetic field*. We will consider Neumann magnetic conditions, that is:

$$\nabla_N^A u = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

where N denotes the inner unit normal. Then, it is well-known that Δ_A is self-adjoint, and admits a discrete spectrum

$$0 \leq \lambda_1(\Delta_A) \leq \lambda_2(\Delta_A) \leq ... \to \infty.$$

The above is a particular case of a more general situation, where $E \to M$ is a complex line bundle with a hermitian connection ∇^E, and where the magnetic Laplacian is defined as $\Delta_E = (\nabla^E)^\star \nabla^E$.

The spectrum of the magnetic Laplacian is very much studied in analysis (see for example [3] and the references therein) and in relation with physics. For *Dirichlet boundary conditions*, lower estimates of its fundamental tone have been worked out, in particular, when Ω is a planar domain and B is the constant magnetic field; that is, when the function $\star B$ is constant on Ω (see for example a Faber–Krahn type inequality in [9] and the recent [12] and the references therein, also for Neumann boundary condition). The case when the potential A is a closed 1-form is particularly interesting from the physical point of view (Aharonov–Bohm effect), and also from the geometric point of view. For Dirichlet boundary conditions, there is a series of papers for domains with a pole, when the pole approaches the boundary (see [1,13] and the references therein). Last but not least, there is a Aharonov–Bohm approach to the question of nodal and minimal partitions, see chapter 8 of [4].

For *Neumann boundary conditions*, we refer in particular to the paper [10], where the authors study the multiplicity and the nodal sets corresponding to the ground state λ_1 for non-simply connected planar domains with harmonic potential (see the discussion below).

Let us also mention the recent article [11] (chapter 7) where the authors establish a *Cheeger type inequality* for λ_1; that is, they find a lower bound for $\lambda_1(\Delta_A)$ in terms of the geometry of Ω and the potential A. In the preprint [8], the authors approach the