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# The Hadamard determinant inequality — Extensions to operators on a Hilbert space

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ABSTRACT

A generalization of classical determinant inequalities like Hadamard's inequality and Fischer's inequality is studied. For a version of the inequalities originally proved by Arveson for positive operators in von Neumann algebras with a tracial state, we give a different proof. We also improve and generalize to the setting of finite von Neumann algebras, some 'Fischer-type' inequalities by Matic for determinants of perturbed positive-definite matrices. In the process, a conceptual framework is established for viewing these inequalities as manifestations of Jensen's inequality in conjunction with the theory of operator monotone and operator convex functions on $[0, \infty)$. We place emphasis on documenting necessary and sufficient conditions for equality to hold.


## 1. Introduction

At their core, the many applications of determinants in mathematical analysis are based on the geometric interpretation of the determinant of a square matrix as the (signed) volume of an $n$-parallelepiped with sides as the column vectors of the matrix.

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For instance, the change-of-variables formula in multidimensional integration involves the determinant of the Jacobian matrix. The study of estimates for the determinant of a matrix in terms of determinants of its principal submatrices is often useful as information about compressions of a matrix $A$ to certain subspaces (*i.e.* $PAP$ for a projection $P$) may be more readily available. An element of $M_n(\mathbb{C})$, the set of complex $n \times n$ matrices, is said to be *positive-semidefinite* if it is Hermitian with non-negative eigenvalues, and *positive-definite* if it is positive-semidefinite with strictly positive eigenvalues. Let $A$ be a positive-definite matrix with $(i,j)$th entry denoted by $a_{ij}$. Hadamard's inequality ([10]) states that the determinant of a positive-definite matrix is less than or equal to the product of the diagonal entries of the matrix *i.e.* $\det A \leq \prod_{i=1}^{n} a_{ii}$. Further, equality holds if and only if $A$ is a diagonal matrix. As a corollary, which is usually referred to by the same name, we get that the absolute value of the determinant of a square matrix is less than or equal to the product of the Euclidean norm of its column vectors (or alternatively, row vectors). In the case of real matrices, the inequality conveys the geometrically intuitive idea that an $n$-parallelepiped with prescribed lengths of sides has largest volume if and only if the sides are mutually orthogonal. An important application of this inequality to the theory of integral equations is in proving convergence results in classical Fredholm theory ([21, section 5.3]). More generally, a similar inequality, known as Fischer's inequality ([8]), holds if one considers the principal diagonal blocks of a positive-definite matrix in block form. Hadamard's inequality is a corollary of Fischer's inequality by considering blocks of size $1 \times 1$.

For $n \in \mathbb{N}$, we denote the indexing set $\{1, 2, \cdots, n\}$ by $\langle n \rangle$. In Fischer's inequality below, for an $n \times n$ matrix $A$ and $\alpha \subseteq \langle n \rangle$, the principal submatrix of $A$ from rows and columns indexed by $\alpha$ is denoted by $A[\alpha]$.

**Theorem 1.1** *(Fischer's inequality). Let $A$ be a positive-definite matrix in $M_n(\mathbb{C})$. Let $\alpha_i \subseteq \langle n \rangle$ for $i \in \langle k \rangle$ such that $\alpha_i \cap \alpha_j = \varnothing$ for $i, j \in \langle k \rangle, i \neq j$. Then*

$$\det(A[\cup_{i=1}^{k} \alpha_i]) \leq \prod_{i=1}^{k} \det(A[\alpha_i])$$

*with equality if and only if $A[\cup_{i=1}^{k} \alpha_i] = P \operatorname{diag}(A[\alpha_1], \cdots, A[\alpha_k]) P^{-1}$ for some permutation matrix $P$.*

Unwrapping the notation in the above theorem, we have that the determinant of a positive-definite matrix (in block form) is less than or equal to the product of the determinants of its principal diagonal blocks, with equality if and only if the entries outside the principal diagonal blocks are all 0. We state an application of this result to information theory. For a multivariate normal random variable $(X_1, X_2, \cdots, X_n)$ with mean $\mathbf{0}$, covariance matrix $\Sigma$ and hence density

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} \exp(-\frac{1}{2} \mathbf{x}^T \Sigma^{-1} \mathbf{x}), \mathbf{x} \in \mathbb{R}^n,$$