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Explicit formulas for $C^{1,1}$ and $C^{1,\omega}_{\text{conv}}$ extensions of 1-jets in Hilbert and superreflexive spaces [☆]

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Given X a Hilbert space, ω a modulus of continuity, E an arbitrary subset of X, and functions $f : E \to \mathbb{R}$, $G : E \to X$, we provide necessary and sufficient conditions for the jet (f, G) to admit an extension $(F, \nabla F)$ with $F : X \to \mathbb{R}$ convex and of class $C^{1,\omega}(X)$, by means of a simple explicit formula. As a consequence of this result, if ω is linear, we show that a variant of this formula provides explicit $C^{1,1}$ extensions of general (not necessarily *convex*) 1-jets satisfying the usual Whitney extension condition, with best possible Lipschitz constants of the gradients of the extensions. Finally, if X is a superreflexive Banach space, we establish similar results for the classes $C^{1,\alpha}_{\text{conv}}(X)$.

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$C^{1,\omega}$ function
Whitney extension theorem

1. Introduction and main results

If C is a subset of $\mathbb{R}^n$ and we are given functions $f : C \to \mathbb{R}$, $G : C \to \mathbb{R}^n$, the $C^{1,1}$ version of the classical Whitney extension theorem (see [28,15,24] for instance) theorem tells us that there exists a function $F \in C^{1,1}(\mathbb{R}^n)$ with $F = f$ on C and $\nabla F = G$ on C if and only if the 1-jet (f, G) satisfies the following property: there exists a constant $M > 0$ such that

$$|f(x) - f(y) - \langle G(y), x - y\rangle| \leq M|x - y|^2, \quad \text{and} \quad |G(x) - G(y)| \leq M|x - y| \qquad (\widetilde{W^{1,1}})$$

for all $x, y \in C$. We can trivially extend (f, G) to the closure $\overline{C}$ of C so that the inequalities $(\widetilde{W^{1,1}})$ hold on $\overline{C}$ with the same constant M. The function F can be explicitly defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \overline{C} \\ \sum_{Q\in\mathcal{Q}} \left(f(x_Q) + \langle G(x_Q), x - x_Q\rangle\right) \varphi_Q(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{C}, \end{cases}$$

where $\mathcal{Q}$ is a family of *Whitney cubes* that cover the complement of the closure $\overline{C}$ of C, $\{\varphi_Q\}_{Q\in\mathcal{Q}}$ is the usual Whitney partition of unity associated to $\mathcal{Q}$, and x_Q is a point of $\overline{C}$ which minimizes the distance of $\overline{C}$ to the cube Q. Recall also that the function F constructed in this way has the property that $\mathrm{Lip}(\nabla F) \leq k(n)M$, where $k(n)$ is a constant depending only on n (but going to infinity as $n \to \infty$), and $\mathrm{Lip}(\nabla F)$ denotes the Lipschitz constant of the gradient ∇F.

In [27,20] it was shown, by very different means, that this $C^{1,1}$ version of the Whitney extension theorem holds true if we replace $\mathbb{R}^n$ with any Hilbert space and, moreover, there is an extension operator $(f, G) \mapsto (F, \nabla F)$ which is minimal, in the following sense. Given a Hilbert space X with norm denoted by $\|\cdot\|$, a subset E of X, and functions $f : E \to \mathbb{R}$, $G : E \to X$, a necessary and sufficient condition for the 1-jet (f, G) to have a $C^{1,1}$ extension $(F, \nabla F)$ to the whole space X is that

$$\Gamma(f, G, E) := \sup_{x,y\in E} \left(\sqrt{A_{x,y}^2 + B_{x,y}^2} + |A_{x,y}|\right) < \infty, \tag{1.1}$$

where

$$A_{x,y} = \frac{2(f(x) - f(y)) + \langle G(x) + G(y), y - x\rangle}{\|x - y\|^2} \quad \text{and}$$

$$B_{x,y} = \frac{\|G(x) - G(y)\|}{\|x - y\|} \quad \text{for all} \quad x, y \in E, x \neq y.$$

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