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Translational absolute continuity and Fourier frames on a sum of singular measures ☆

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ABSTRACT

A finite Borel measure μ in $\mathbb{R}^d$ is called a frame-spectral measure if it admits an exponential frame (or Fourier frame) for $L^2(\mu)$. It has been conjectured that a frame-spectral measure must be translationally absolutely continuous, which is a criterion describing the local uniformity of a measure on its support. In this paper, we show that if any measures ν and λ without atoms whose supports form a packing pair, then $\nu * \lambda + \delta_t * \nu$ is translationally singular and it does not admit any Fourier frame. In particular, we show that the sum of one-fourth and one-sixteenth Cantor measure $\mu_4 + \mu_{16}$ does not admit any Fourier frame. We also interpolate the mixed-type frame-spectral measures studied by Lev and the measure we studied. In doing so, we demonstrate a discontinuity behavior: For any anticlockwise rotation mapping R_θ with $\theta \neq \pm\pi/2$, the two-dimensional measure $\rho_\theta(\cdot) := (\mu_4 \times \delta_0)(\cdot) + (\delta_0 \times \mu_{16})(R_\theta^{-1}\cdot)$, supported on the union of x-axis and $y = (\cot\theta)x$, always admit a Fourier frame. Furthermore, we can find $\{e^{2\pi i\langle\lambda,x\rangle}\}_{\lambda\in\Lambda_\theta}$ such that it forms a Fourier frame for ρ_θ with frame bounds independent of θ. Nonetheless, $\rho_{\pm\pi/2}$ does not admit any Fourier frame.

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1. Introduction

Definition 1.1. Let μ be a finite Borel measure on $\mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ denote the standard inner product. We say that μ is a *frame-spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ such that a system of exponential functions $E(\Lambda) := \{e_\lambda(x) : \lambda \in \Lambda\}$, where $e_\lambda(x) = e^{2\pi i\langle\lambda,x\rangle}$, forms a *Fourier frame* for $L^2(\mu)$, i.e., there exist two constants A, B such that $0 < A \le B < \infty$ with

$$A\|f\|^2_{L^2(\mu)} \le \sum_{\lambda\in\Lambda} |\int f(x)e^{2\pi i\langle\lambda,x\rangle}d\mu(x)|^2 \le B\|f\|^2_{L^2(\mu)} \tag{1.1}$$

for any $f \in L^2(\mu)$. Whenever such Λ exists, Λ is called a *frame spectrum* for μ. If only the upper bound holds, $E(\Lambda)$ is called *a Bessel sequence* for μ. We say that μ is a *spectral measure* and Λ is a *spectrum* for μ if the system $E(\Lambda)$ forms an orthonormal basis for $L^2(\mu)$.

The notion of Fourier Frames is a natural generalization of an exponential orthonormal basis, which is a spanning set with possible redundancies in the representation. It was first introduced by Duffin and Schaeffer [5] in the context of non-harmonic Fourier series. Frames nowadays have wide range of applications in signal transmission and reconstruction. The theory of Fourier frames have also been studied in complex analysis since it is equivalent to having a stable sampling in Paley–Wiener spaces [26]. For more general background of frame theory, readers may refer to [1].

One of the basic questions in frame theory is to classify when a measure μ is spectral or frame-spectral. The origin of this question dates back to Fuglede [15] who initiated a study to determine when a measurable set Ω admits an exponential orthonormal basis for $L^2(\Omega)$ and proposed his famous spectral set conjecture. Although the conjecture was disproved in $\mathbb{R}^d, d \ge 3$ by Tao, Kolountzakis and Matolcsi [31,19,20], the conjecture has led people to research extensively the property leading a set or a measure to be spectral or frame-spectral. Particularly, the study of singular spectral measures is possible, difficult but compelling [18,21,6,30]. One of the major properties of frame-spectral measures that has been observed in several earlier work ([22,8]) and was rigorously formulated in [7], is a notion of "translational uniformity".

Given a finite Borel measure μ, the *closed support* of μ is the smallest closed set K such that μ has full measure (i.e. $\mu(K) = \mu(\mathbb{R}^d)$). We will denote it by K_μ throughout the paper. We say that a Borel set X is a *Borel support* of μ if $X \subset K_\mu$ and $\mu(X) = \mu(K_\mu)$. Note that the closed support of μ is unique, but Borel support is not unique. We have the following definition, which refines the definition proposed by Dutkay and the second-named author:

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