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Regularity in L_p Sobolev spaces of solutions to fractional heat equations

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A B S T R A C T

This work contributes in two areas, with sharp results, to the current investigation of regularity of solutions of heat equations with a nonlocal operator P:

$$Pu + \partial_t u = f(x,t), \text{ for } x \in \Omega \subset \mathbb{R}^n,\ t \in I \subset \mathbb{R}. \qquad (*)$$

1) For strongly elliptic pseudodifferential operators (ψdo's) P on $\mathbb{R}^n$ of order $d \in \mathbb{R}_+$, a symbol calculus on $\mathbb{R}^{n+1}$ is introduced that allows showing optimal regularity results, globally over $\mathbb{R}^{n+1}$ and locally over $\Omega \times I$:

$$f \in H^{(s,s/d)}_{p,\mathrm{loc}}(\Omega \times I) \implies u \in H^{(s+d,s/d+1)}_{p,\mathrm{loc}}(\Omega \times I),$$

for $s \in \mathbb{R}$, $1 < p < \infty$. The $H^{(s,s/d)}_p$ are anisotropic Sobolev spaces of Bessel-potential type, and there is a similar result for Besov spaces.
2) Let Ω be smooth bounded, and let P equal $(-\Delta)^a$ ($0 < a < 1$), or its generalizations to singular integral operators with regular kernels, generating stable Lévy processes. With the Dirichlet condition $\operatorname{supp} u \subset \overline{\Omega}$, the initial condition $u|_{t=0} = 0$, and $f \in L_p(\Omega \times I)$, (*) has a unique solution $u \in L_p(I; H^{a(2a)}_p(\overline{\Omega}))$ with $\partial_t u \in L_p(\Omega \times I)$. Here $H^{a(2a)}_p(\overline{\Omega}) = \dot H^{2a}_p(\overline{\Omega})$ if $a < 1/p$, and is contained in $\dot H^{2a-\varepsilon}_p(\overline{\Omega})$ if $a = 1/p$, but contains nontrivial elements from $d^a \overline{H}^a_p(\Omega)$ if $a > 1/p$

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(where $d(x) = \text{dist}(x, \partial\Omega)$). The interior regularity of u is lifted when f is more smooth.

0. Introduction

There is currently a great interest for evolution problems (heat equations)

$$Pu(x,t) + \partial_t u(x,t) = f(x,t) \text{ on } \Omega \times I,\ \Omega \text{ open } \subset \mathbb{R}^n,\ I =]0,T[\,, \tag{0.1}$$

where P is a nonlocal operator, as for example the fractional Laplacian $(-\Delta)^a$ $(0 < a < 1)$ or other pseudodifferential operators (ψdo's) or singular integral operators. For differential operators P, there are classical treatises such as Ladyzhenskaya, Solonnikov and Uraltseva [30] with L_p-methods, Lions and Magenes [33] with L_2-methods, Friedman [11] with L_2 semigroup methods, and numerous more recent studies. Motivated by the linearized Navier–Stokes problem, which can be reduced to the form (0.1) with nonlocal ingredients, the author jointly with Solonnikov treated such problems in [24] (for L_2-spaces) and [17] (for L_p-spaces). In those papers, the operator P fits into the Boutet de Monvel calculus [5,15,16,19], and is necessarily of integer order.

This does not cover fractional order operators, and the present paper aims to find techniques to handle (0.1) in fractional cases. Firstly, we treat ψdo's without boundary conditions in Sections 2 and 3, where we introduce a systematic calculus that allows showing regularity results globally in $\mathbb{R}^{n+1}$, and locally in arbitrary open subset $\Sigma \subset \mathbb{R}^{n+1}$, in terms of anisotropic function spaces described in detail in Appendix A:

Theorem 0.1. *Let P be a classical strongly elliptic ψdo $P = \mathrm{OP}(p(x,\xi))$ on $\mathbb{R}^n$ of order $d \in \mathbb{R}_+$. Let $s \in \mathbb{R}$, $1 < p < \infty$. Then $P + \partial_t$ maps $H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R}) \to H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$.*

1° Let $u \in H_p^{(r,r/d)}(\mathbb{R}^n \times \mathbb{R})$ for some large negative r (this holds in particular if $u \in \mathcal{E}'(\mathbb{R}^{n+1})$ or e.g. $L_p(\mathbb{R}; \mathcal{E}'(\mathbb{R}^n))$). Then

$$(P + \partial_t)u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) \implies u \in H_p^{(s+d,s/d+1)}(\mathbb{R}^n \times \mathbb{R}). \tag{0.2}$$

2° Let Σ be an open subset of $\mathbb{R}^{n+1}$, and let $u \in H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R})$. Then

$$(P + \partial_t)u|_\Sigma \in H_{p,\mathrm{loc}}^{(s,s/d)}(\Sigma) \implies u \in H_{p,\mathrm{loc}}^{(s+d,s/d+1)}(\Sigma). \tag{0.3}$$

The analogous result holds in Besov-spaces $B_p^{(s,s/d)}$, and there is also a result in anisotropic Hölder spaces that can be derived from (0.3) by letting $p \to \infty$.