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On Green functions of second-order elliptic operators on Riemannian manifolds: The critical case

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A B S T R A C T

Let P be a second-order, linear, elliptic operator with real coefficients which is defined on a noncompact and connected Riemannian manifold M. It is well known that the equation $Pu = 0$ in M admits a positive supersolution which is not a solution if and only if P admits a unique positive minimal Green function on M, and in this case, P is said to be *subcritical* in M. If P does not admit a positive Green function but admits a global positive (super)solution, then such a solution is called a *ground state* of P in M, and P is said to be *critical* in M.
We prove for a critical operator P in M, the existence of a Green function which is dominated above by the ground state of P away from the singularity. Moreover, in a certain class of Green functions, such a Green function is unique, up to an addition of a product of the ground states of P and $P^\star$. Under some further assumptions, we describe the behavior at infinity of such a Green function. This result extends and sharpens the celebrated result of P. Li and L.-F. Tam concerning the existence of a *symmetric* Green function for the Laplace–Beltrami operator on a smooth and *complete* Riemannian manifold M.

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1. Introduction

Let M be a noncompact and connected manifold of dimension $N \geq 2$ and of class C^2. We assume that ν is a positive measure on M, satisfying $\mathrm{d}\nu = f\,\mathrm{dvol}$, where f is a strictly positive function and vol is the volume form of M. On M we consider a second-order elliptic operator P with real coefficients which (in any coordinate system $(U; x_1, \ldots, x_N)$) is of the divergence form

$$Pu := -\mathrm{div}\left[\left(A(x)\nabla u + u\tilde{b}(x)\right)\right] + b(x)\cdot\nabla u + c(x)u. \tag{1.1}$$

Here, the minus divergence is the formal adjoint of the gradient with respect to the measure ν. We assume that for every $x \in \Omega$ the matrix $A(x) := \left[a^{ij}(x)\right]$ is symmetric and that the real quadratic form

$$\xi \cdot A(x)\xi := \sum_{i,j=1}^{N} \xi_i a^{ij}(x)\xi_j \qquad \xi \in \mathbb{R}^N \tag{1.2}$$

is positive definite. Moreover, throughout the paper it is assumed that P is locally uniformly elliptic, and that locally, the coefficients of P are sufficiently regular in M such that standard elliptic (local) regularity results hold true. Our results hold for example when A and f are locally Hölder continuous, b, $\tilde{b}$ are Borel measurable vector fields in M of class $L^p_{\mathrm{loc}}(M)$, and $c \in L^{p/2}_{\mathrm{loc}}(M)$ for some $p > N$. In fact, we need to assume further local regularity on the coefficients that guarantee the existence of the limit

$$\lim_{x\to x_0} \frac{u(x)}{v(x)}\,, \tag{1.3}$$

where u and v are positive solutions of the equation $Pu = 0$ in a punctured neighborhood of any $x_0 \in M$, and the limit might be ∞ (for sufficient conditions that guarantee it, see for example [9] and references therein).

The formal adjoint P^* of the operator P is defined on its natural space $L^2(M, \mathrm{d}\nu)$. When P is in divergence form (1.1) and $b = \tilde{b}$, the operator

$$Pu = -\mathrm{div}\left[\left(A\nabla u + ub\right)\right] + b\cdot\nabla u + cu,$$

is *symmetric* in the space $L^2(M, \mathrm{d}\nu)$. Throughout the paper, we call this setting the *symmetric case*.

By a solution v of the equation $Pu = 0$ in a domain $\Omega \subset M$, we mean $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ that satisfies the equation $Pu = 0$ in Ω in the *weak sense*. Subsolutions and supersolutions are defined similarly. We denote the cone of all positive solutions of the equation $Pu = 0$ in Ω by $\mathcal{C}_P(\Omega)$. We say that P is *nonnegative in* Ω (and denote it by $P \geq 0$ in Ω) if $\mathcal{C}_P(\Omega) \neq \emptyset$. We recall that in the symmetric case, by the Allegretto–Piepenbrink theorem,