Journal of Functional Analysis 274 (2018) 2163–2204

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

A probabilistic approach to spectral analysis of growth-fragmentation equations

Jean Bertoin [a], Alexander R. Watson [b]

[a] *Institute of Mathematics, University of Zurich, Switzerland*
[b] *School of Mathematics, University of Manchester, UK*

A R T I C L E   I N F O

Article history:
Received 17 January 2017
Accepted 18 January 2018
Available online 5 February 2018
Communicated by M. Hairer

MSC:
35Q92
47D06
45K05
47G20
60G51

Keywords:
Growth-fragmentation equation
Spectral analysis
Malthus exponent
Feynman–Kac formula

A B S T R A C T

The growth-fragmentation equation describes a system of growing and dividing particles, and arises in models of cell division, protein polymerisation and even telecommunications protocols. Several important questions about the equation concern the asymptotic behaviour of solutions at large times: at what rate do they converge to zero or infinity, and what does the asymptotic profile of the solutions look like? Does the rescaled solution converge to its asymptotic profile at an exponential speed? These questions have traditionally been studied using analytic techniques such as entropy methods or splitting of operators. In this work, we present a probabilistic approach: we use a Feynman–Kac formula to relate the solution of the growth-fragmentation equation to the semigroup of a Markov process, and characterise the rate of decay or growth in terms of this process. We then identify the Malthus exponent and the asymptotic profile in terms of a related Markov process, and give a spectral interpretation in terms of the growth-fragmentation operator and its dual.

E-mail address: jean.bertoin@math.uzh.ch (J. Bertoin).

https://doi.org/10.1016/j.jfa.2018.01.014

1. Introduction

This work studies the asymptotic behaviour of solutions to the growth-fragmentation equation using probabilistic methods. The growth-fragmentation arises from mathematical models of biological phenomena such as cell division [37, §4] and protein polymerization [21], as well as in telecommunications [28]. The equation describes the evolution of the density $u_t(x)$ of particles of mass $x > 0$ at time $t \geq 0$, in a system whose dynamics are given as follows. Each particle grows at a certain rate depending on its mass and experiences 'dislocation events', again at a rate depending on its mass. At each such event, it splits into smaller particles in such a way that the total mass is conserved. The growth-fragmentation equation is a partial integro-differential equation and can be expressed in the form

$$\partial_t u_t(x) + \partial_x (c(x) u_t(x)) = \int_x^\infty u_t(y) k(y,x) \mathrm{d}y - K(x) u_t(x), \tag{1}$$

where $c\colon (0,\infty) \to (0,\infty)$ is a continuous positive function specifying the growth rate, $k\colon (0,\infty) \times (0,\infty) \to \mathbb{R}_+$ is a so-called fragmentation kernel, and the initial condition u_0 is prescribed. In words, $k(y,x)$ represents the rate at which a particle with size x appears as the result of the dislocation of a particle with mass $y > x$. More precisely, the fragmentation kernel fulfils

$$k(x,y) = 0 \text{ for } y > x, \text{ and } \int_0^x y k(x,y) \mathrm{d}y = xK(x). \tag{2}$$

The first requirement stipulates that after the dislocation of a particle, only particles with smaller masses can arise. The second reflects the conservation of mass at dislocation events, and gives the interpretation of $K(x)$ as the total rate of dislocation of particles with size x.

This equation has been studied extensively over many years. A good introduction to growth-fragmentation equations and related equations in biology can be found in the monographs of Perthame [37] and Engel and Nagel [17], and a major issue concerns the asymptotic behaviour of solutions u_t. Besides being interesting from the perspective of the differential equation, this asymptotic behaviour tells us something about the fitness of a related stochastic cell model [11,12]. Typically, one wishes to find a constant $\lambda \in \mathbb{R}$, the *Malthus exponent*, for which $\mathrm{e}^{-\lambda t} u_t$ converges, in some suitable space, to a so-called *asymptotic profile* v. Ideally, we would also like to have some information about the *rate of convergence*; that is, we would like to ensure the existence of some $\beta > 0$ with the property that $\mathrm{e}^{\beta t}(\mathrm{e}^{-\lambda t} u_t - v)$ converges to zero.

For such questions, a key step in finding λ is the spectral analysis of the growth-fragmentation operator