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Weighted jump and variational inequalities for rough operators ☆

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In this paper, we systematically study weighted jump and variational inequalities for rough operators. More precisely, we show some weighted jump and variational inequalities for the families $\mathcal{T} := \{T_\varepsilon\}_{\varepsilon>0}$ of truncated singular integrals and $\mathcal{M}_\Omega := \{M_{\Omega,t}\}_{t>0}$ of averaging operators with rough kernels, which are defined respectively by

$$T_\varepsilon f(x) = \int\limits_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y)dy$$

and

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$$M_{\Omega,t}f(x) = \frac{1}{t^n} \int\limits_{|y|<t} \Omega(y')f(x-y)dy,$$

where the kernel Ω belongs to $L^q(\mathbf{S}^{n-1})$ for $q > 1$.

1. Introduction

The jump and variational inequalities have been the subject of many recent articles in probability, ergodic theory and harmonic analysis. To present related results in a precise way, let us fix some notations. Let $I \subset \mathbb{R}$ be an interval. In the later use, I is either $\mathbb{R}_+$ or a dyadic interval. Given a family of complex numbers $\mathfrak{a} = \{a_t : t \in I\}$ and $\rho \geq 1$, the ρ-variation norm of the family $\mathfrak{a}$ is defined by

$$\|\mathfrak{a}\|_{V_\rho} = \sup \big(|a_{t_0}| + \sum_{k\geq 1} |a_{t_k} - a_{t_{k-1}}|^\rho\big)^{\frac{1}{\rho}}, \tag{1.1}$$

where the supremum runs over all finite increasing sequences $\{t_k : k \geq 0\} \subset I$. It is trivial that

$$\|\mathfrak{a}\|_{L^\infty(\mathbb{R})} := \sup_{t\in\mathbb{R}} |a_t| \leq \|\mathfrak{a}\|_{V_\rho} \quad \text{for} \quad \rho \geq 1. \tag{1.2}$$

Via the definition (1.1), one may define the strong ρ-variation function $V_\rho(\mathcal{F})$ of a family $\mathcal{F}$ of functions. Given a family of Lebesgue measurable functions $\mathcal{F} = \{F_t : t \in \mathbb{R}\}$ defined on $\mathbb{R}^n$, for any fixed x in $\mathbb{R}^n$, the value of the strong ρ-variation function $V_\rho(\mathcal{F})$ of the family $\mathcal{F}$ at x is defined by

$$V_\rho(\mathcal{F})(x) = \|\{F_t(x)\}_{t\in\mathbb{R}}\|_{V_\rho}, \quad \rho \geq 1. \tag{1.3}$$

Suppose $\mathcal{A} = \{A_t\}_{t>0}$ is a family of operators on $L^p(\mathbb{R}^n)\,(1 \leq p \leq \infty)$. The strong ρ-variation operator is simply defined as

$$V_\rho(\mathcal{A}f)(x) = \|\{A_t(f)(x)\}_{t>0}\|_{V_\rho}, \quad \forall f \in L^p(\mathbb{R}^n).$$

It is easy to observe from the definition of ρ-variation norm that for any x if $V_\rho(\mathcal{A}f)(x) < \infty$, then $\{A_t(f)(x)\}_{t>0}$ converges when $t \to 0$ or $t \to \infty$. In particular, if $V_\rho(\mathcal{A}f)$ belongs to some function spaces such as $L^p(\mathbb{R}^n)$ or $L^{p,\infty}(\mathbb{R}^n)$, then the sequence converges almost everywhere without any additional condition. This is why the mapping property of the strong ρ-variation operator is so interesting in ergodic theory and harmonic analysis. Also, by (1.2), for any $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have