ARTICLE IN PRESS YJFAN:7952

Journal of Functional Analysis ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

The Schrödinger equation with spatial white noise: The average wave function

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A R T I C L E I N F O

Article history:
Received 5 June 2017
Accepted 25 January 2018
Available online xxxx
Communicated by M. Hairer

Keywords:
Random Schrödinger equation
Renormalization
Intersection local time

A B S T R A C T

We prove a representation for the average wave function of the Schrödinger equation with a white noise potential in $d = 1, 2$, in terms of the renormalized self-intersection local time of a Brownian motion.

1. Introduction

We consider the Schrödinger equation with a large, highly oscillatory random potential

$$i\partial_t \psi_\varepsilon + \frac{1}{2}\Delta\psi_\varepsilon - V_\varepsilon(x)\psi_\varepsilon = 0, \quad \psi_\varepsilon(0, x) = \phi_0(x), \quad x \in \mathbb{R}^d, \tag{1.1}$$

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https://doi.org/10.1016/j.jfa.2018.01.015
0022-1236/

and the initial condition $\phi_0(x)$ that is a compactly supported C^∞ function. The random potential is a microscopically smoothed version of a spatial white noise:

$$V_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} V(\frac{x}{\varepsilon}).$$

Here, V is a stationary, zero-mean and isotropic Gaussian random field over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the expectation denoted by $\mathbb{E}$. If the two-point correlation function

$$R(x) := \mathbb{E}[V(x+y)V(y)] = \rho(|x|), \quad x, y \in \mathbb{R}^d, \tag{1.2}$$

decays sufficiently fast, then $V_\varepsilon(x)$ converges to a spatial white noise $\dot{W}(x)$.

In $d = 2$, this problem was analyzed on a torus $\mathbb{T}^2$ in [6] and the whole space $\mathbb{R}^2$ in [5]. The solution of (1.1) acquires a large phase by $t \sim O(1)$, and the main result of [6] is that the adjusted solution

$$\phi_\varepsilon(t, x) = \psi_\varepsilon(t, x) e^{-iC_\varepsilon t},$$

that satisfies

$$i\partial_t \phi_\varepsilon + \frac{1}{2}\Delta\phi_\varepsilon - (V_\varepsilon(x) + C_\varepsilon)\phi_\varepsilon = 0, \quad \phi_\varepsilon(0, x) = \phi_0(x), \quad x \in \mathbb{R}^d, \tag{1.3}$$

with $C_\varepsilon \sim \log \varepsilon^{-1}$, converges to the solution of the stochastic PDE that can be formally written as

$$i\partial_t \phi_{\text{spde}} + \frac{1}{2}\Delta\phi_{\text{spde}} - \dot{W}(x) \cdot \phi_{\text{spde}} = 0. \tag{1.4}$$

The approach is based on a change of variable used in [8], together with the mass and energy conservations, and also applies to nonlinear equations. By analyzing the Anderson Hamiltonian

$$-\frac{1}{2}\Delta + V_\varepsilon(x) + C_\varepsilon$$

with the paracontrolled calculus, a spectral theory has been established in [1], which also gives a meaning to the solution to (1.4) on $\mathbb{T}^2$.

When $d = 1$, no renormalization is needed and $C_\varepsilon = 0$. It has been proved in [18] that the solution ϕ_ε of (1.3) converges to a solution to (1.4), defined as an infinite series of iterated Stratonovich integrals.

Unfortunately, the information on the limit from the above considerations is rather implicit. Our goal here is to understand some of the properties of the solution to (1.3), in a more direct way. In particular, we establish a representation of $\lim_{\varepsilon\to 0} \mathbb{E}[\widehat{\phi}_\varepsilon]$ in $d = 1, 2$,