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An elliptic theory of indicial weights and applications to non-linear geometry problems

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ABSTRACT

Given an elliptic operator P on a non-compact manifold (with proper asymptotic conditions), there is a discrete set of numbers called indicial roots. It's known that P is Fredholm between weighted Sobolev spaces if and only if the weight is not indicial. We show that an elliptic theory exists even when the weight is indicial. We also discuss some simple applications to Yang–Mills theory and minimal surfaces.

1. Introduction

1.1. The theory

The elliptic theories based on weighted Sobolev (Schauder) spaces usually concern a discrete set of real numbers. If a number is in the set, we say that it is indicial (or is an indicial root). A classical fact says that on a non-compact complete manifold, an elliptic operator (with proper asymptotic conditions) is Fredholm between weighted Sobolev spaces if and only if the weight is not indicial. For earlier pioneering work, please see [9], [10], and [13]. For more recent work, please see [12].

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Following elementary ideas, we show that there is an elliptic theory even if the weight is indicial: first, we add polynomial weights {compare (6) to [10, (1.3)]} to refine the space; second, we consider graph norms with respect to the model operator [see (2)].

In this note we only consider first and second-order operators modelled on the following.

Definition 1.1. Let Y be a $(n-1)$-dimensional Riemannian manifold without boundary (which does not have to be connected). Let E, F be smooth vector-bundles over Y equipped with smooth Hermitian metrics. Given arbitrary bundle isomorphisms $\sigma_1 : E \to F$, $\sigma_2 : E \to E$, we say that an operator P^0 is TID (translation-invariant and diagonal) if

$$P^0 = \sigma_1(-B_{P^0} - a_1\frac{\partial}{\partial t} + a_2\frac{\partial^2}{\partial t^2})\sigma_2 \text{ and the following holds.} \tag{1}$$

- $a_2 = 0$ or 1. $a_1 = -1$ when $a_2 = 0$ (always achievable by normalization).
- When $a_2 = 0$, B_{P^0} is a first-order self-adjoint elliptic differential operator $C^\infty(Y,E) \to C^\infty(Y,E)$. When $a_2 = 1$, B_{P^0} is second-order, simple, elliptic, and self-adjoint $C^\infty(Y,E) \to C^\infty(Y,E)$ (see Definition 2.1).

Remark 1.2. For any TID operator P^0, $SpecB_{P^0}$ is real and discrete. Moreover, there is a complete eigen-basis of B_{P^0}.

Definition 1.3. Let P^0 be TID, and (β,Λ) be a pair of real numbers such that $\Lambda \in Spec(B_{P^0})$. When P^0 is first-order, we say that (β,Λ) is P^0-indicial if $\beta = \Lambda$. When P^0 is second-order, we say that (β,Λ) is P^0-indicial if one of the following holds.

1. $\beta \neq \frac{a_1}{2}$ and $\Lambda - \beta^2 + a_1\beta = 0$,
2. $\Lambda \leq -\frac{a_1^2}{4}$ and $\beta = \frac{a_1}{2}$.

In the second case above, we say that (β,Λ) is P^0-super indicial. We say that β is P^0-indicial (super indicial) if there is a $\Lambda \in SpecB_{P^0}$ such that (β,Λ) is P^0-indicial (super indicial). This is consistent with the "$\mathfrak{D}_A$" in [10, page 417], translated to our setting.

Let N be a complete Riemannian manifold with finite many cylindrical ends, we consider asymptotically TID operators $P : C^\infty(N,E) \to C^\infty(N,F)$. This class should include most of the Dirac and Laplace-type operators in geometry. Let m_0 denote the order of P. In this article, we assume $m_0 = 1$ or 2. In the setting as Theorem 1.4,

$$\begin{aligned} &P:\ \widehat{W}^{k+m_0,p}_{-\beta,\gamma,b-1}(N,E) \longrightarrow W^{k,p}_{-\beta,\gamma,b}(N,F) && (\square|^{Sobolev,p}_{-\beta,\gamma,b}) \text{ (see Definition 2.3, 2.10} \\ &P:\ \widehat{C}^{k+m_0,\alpha}_{-\beta,\gamma,b-1}(N,E) \longrightarrow C^{k,\alpha}_{-\beta,\gamma,b}(N,F) && (\square|^{Schauder}_{-\beta,\gamma,b}) \quad \text{for the norms),} \end{aligned} \tag{2}$$