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Topological bounds for Fourier coefficients and applications to torsion

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ABSTRACT

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain in the plane and consider

$$\begin{aligned} -\Delta u &= 1 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

If u assumes its maximum in $x_0 \in \Omega$, then the eccentricity of level sets close to the maximum is determined by the Hessian $D^2u(x_0)$. We prove that $D^2u(x_0)$ is negative definite and give a quantitative bound on the spectral gap

$$\lambda_{\max}\left(D^2u(x_0)\right) \leq -c_1 \exp\left(-c_2 \frac{\operatorname{diam}(\Omega)}{\operatorname{inrad}(\Omega)}\right)$$
for universal $c_1, c_2 > 0$.

This is sharp up to constants. The proof is based on a new lower bound for Fourier coefficients whose proof has a topological component: if $f : \mathbb{T} \to \mathbb{R}$ is continuous and has n sign changes, then

$$\sum_{k=0}^{n/2} |\langle f, \sin kx\rangle| + |\langle f, \cos kx\rangle| \gtrsim_n \frac{|f\|_{L^1(\mathbb{T})}^{n+1}}{\|f\|_{L^\infty(\mathbb{T})}^{n}}.$$

This statement immediately implies estimates on higher derivatives of harmonic functions u in the unit ball: if

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u is very flat in the origin, then the boundary function $u(\cos t, \sin t) : \mathbb{T} \rightarrow \mathbb{R}$ has to have either large amplitude or many roots. It also implies that the solution of the heat equation starting with $f : \mathbb{T} \rightarrow \mathbb{R}$ cannot decay faster than $\sim \exp(-(\#\text{sign changes})^2 t/4)$.

1. Introduction

1.1. Introduction

When studying the solution of elliptic equations on planar domains, there is a clear tendency for level sets to become more elliptical and regular. Different aspects of this phenomenon have been studied for a long time and there are many classical results [4,6,5,7,10,13,18,19,21,24,26]; these results are usually centered around the question whether the level sets are convex or remain convex if the underlying domain is convex (see Fig. 1). Despite a lot of work, the subject is still not thoroughly understood. A very simple question is whether the solution of $-\Delta u = f(u)$ on a convex domain Ω with Dirichlet boundary conditions $\partial\Omega$ always has convex level sets – this is fairly natural to assume (see e.g. P.-L. Lions [17]) and was only very recently answered in the negative by Hamel, Nadirashvili and Sire [11].

2. Main results

2.1. The torsion function

We consider, for bounded, convex $\Omega \subset \mathbb{R}^2$, the torsion function

$$\begin{aligned} -\Delta u &= 1 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

The torsion function is perhaps *the* classical object in the study of level sets of elliptic equations. Makar-Limanov [20] proved that $\sqrt{u}$ is concave on planar convex domains:

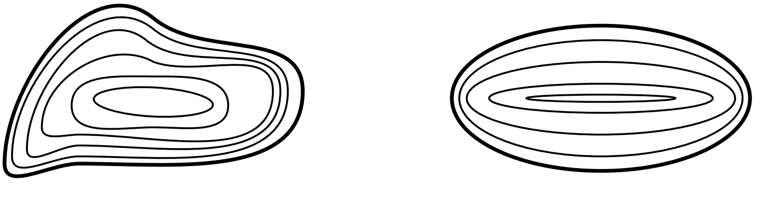

Fig. 1. A cartoon picture of what 'typical level sets' of $-\Delta u = f(u)$ could look like (left) and how they probably will not look like (right).