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Schrödinger operators with negative potentials and Lane–Emden densities

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A B S T R A C T

We consider the Schrödinger operator $-\Delta + V$ for negative potentials V, on open sets with positive first eigenvalue of the Dirichlet–Laplacian. We show that the spectrum of $-\Delta+V$ is positive, provided that V is greater than a negative multiple of the logarithmic gradient of the solution to the Lane–Emden equation $-\Delta u = u^{q-1}$ (for some $1 \leq q < 2$). In this case, the ground state energy of $-\Delta + V$ is greater than the first eigenvalue of the Dirichlet–Laplacian, up to an explicit multiplicative factor. This is achieved by means of suitable Hardy-type inequalities, that we prove in this paper.

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Contents

1. Introduction

1.1. Foreword

Let $V \in L^2_{\rm loc}(\mathbb{R}^N)$ be a real-valued potential such that $V \leq 0$ and let us consider the *Schrödinger operator* $\mathcal{H}_V := -\Delta + V$, acting on the domain

$$\mathfrak{D}(\mathcal{H}_V) := H^2(\mathbb{R}^N) \cap \{u \in L^2(\mathbb{R}^N) \,:\, V\,u \in L^2(\mathbb{R}^N)\}.$$

Observe that the hypothesis $V \in L^2_{\rm loc}(\mathbb{R}^N)$ entails the inclusion

$$C^\infty_0(\mathbb{R}^N) \subset \mathfrak{D}(\mathcal{H}_V),$$

thus $\mathfrak{D}(\mathcal{H}_V)$ is dense in $L^2(\mathbb{R}^N)$. The operator $\mathcal{H}_V : \mathfrak{D}(\mathcal{H}_V) \to L^2(\mathbb{R}^N)$ is symmetric and self-adjoint as well, thanks to the fact that V is real-valued (see [15, Example, p. 68]). The *spectrum* of $\mathcal{H}_V$ is the set

$$\sigma(\mathcal{H}_V) = \mathbb{R} \setminus \rho(\mathcal{H}_V),$$

where $\rho(\mathcal{H}_V)$ is the *resolvent set of* $\mathcal{H}_V$, defined as the collection of real numbers λ such that $\mathcal{H}_V - \lambda$ is bijective and its inverse is a bounded linear operator.

A distinguished subset of $\sigma(\mathcal{H}_V)$ is given by the collection of those λ such that the kernel of $\mathcal{H}_V - \lambda$ is nontrivial. In this case, the *stationary Schrödinger equation*

$$\mathcal{H}_V\, u = \lambda\, u, \tag{1.1}$$