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# Schrödinger operators with negative potentials and Lane–Emden densities



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## ABSTRACT

We consider the Schrödinger operator  $-\Delta + V$  for negative potentials  $V$ , on open sets with positive first eigenvalue of the Dirichlet–Laplacian. We show that the spectrum of  $-\Delta + V$  is positive, provided that  $V$  is greater than a negative multiple of the logarithmic gradient of the solution to the Lane–Emden equation  $-\Delta u = u^{q-1}$  (for some  $1 \leq q < 2$ ). In this case, the ground state energy of  $-\Delta + V$  is greater than the first eigenvalue of the Dirichlet–Laplacian, up to an explicit multiplicative factor. This is achieved by means of suitable Hardy-type inequalities, that we prove in this paper.

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**1. Introduction**

*1.1. Foreword*

Let  $V \in L^2_{\text{loc}}(\mathbb{R}^N)$  be a real-valued potential such that  $V \leq 0$  and let us consider the Schrödinger operator  $\mathcal{H}_V := -\Delta + V$ , acting on the domain

$$\mathfrak{D}(\mathcal{H}_V) := H^2(\mathbb{R}^N) \cap \{u \in L^2(\mathbb{R}^N) : Vu \in L^2(\mathbb{R}^N)\}.$$

Observe that the hypothesis  $V \in L^2_{\text{loc}}(\mathbb{R}^N)$  entails the inclusion

$$C^\infty_0(\mathbb{R}^N) \subset \mathfrak{D}(\mathcal{H}_V),$$

thus  $\mathfrak{D}(\mathcal{H}_V)$  is dense in  $L^2(\mathbb{R}^N)$ . The operator  $\mathcal{H}_V : \mathfrak{D}(\mathcal{H}_V) \rightarrow L^2(\mathbb{R}^N)$  is symmetric and self-adjoint as well, thanks to the fact that  $V$  is real-valued (see [15, Example, p. 68]). The spectrum of  $\mathcal{H}_V$  is the set

$$\sigma(\mathcal{H}_V) = \mathbb{R} \setminus \rho(\mathcal{H}_V),$$

where  $\rho(\mathcal{H}_V)$  is the *resolvent set* of  $\mathcal{H}_V$ , defined as the collection of real numbers  $\lambda$  such that  $\mathcal{H}_V - \lambda$  is bijective and its inverse is a bounded linear operator.

A distinguished subset of  $\sigma(\mathcal{H}_V)$  is given by the collection of those  $\lambda$  such that the kernel of  $\mathcal{H}_V - \lambda$  is nontrivial. In this case, the *stationary Schrödinger equation*

$$\mathcal{H}_V u = \lambda u, \tag{1.1}$$

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