Journal of Functional Analysis 274 (2018) 1255–1275

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Improved Cotlar's inequality in the context of local Tb theorems

Henri Martikainen [a,*,1], Mihalis Mourgoglou [b,2,3], Xavier Tolsa [c,d,4]

[a] Department of Mathematics and Statistics, University of Helsinki, P.O.B. 68, FI-00014 University of Helsinki, Finland
[b] Departament de Matemàtiques, Universitat Autònoma de Barcelona and Centre de Reserca Matemàtica, Edifici C Facultat de Ciències, 08193 Bellaterra (Barcelona), Catalonia
[c] ICREA, Passeig Lluís Companys 23, 08010 Barcelona, Catalonia
[d] Departament de Matemàtiques and BGSMath, Universitat Autònoma de Barcelona, Edifici C Facultat de Ciències, 08193 Bellaterra (Barcelona), Catalonia

A R T I C L E   I N F O

Article history:
Received 20 January 2016
Accepted 28 December 2017
Available online 5 January 2018
Communicated by L. Saloff-Coste

MSC:
42B20

Keywords:
Non-homogeneous analysis
Local Tb theorems
Cotlar's inequality

A B S T R A C T

In the context of local Tb theorems with L^p testing conditions we prove an enhanced Cotlar's inequality. This is related to the problem of removing the so called buffer assumption of Hytönen–Nazarov, which is the final barrier for the full solution of S. Hofmann's problem. We also investigate the problem of extending the Hytönen–Nazarov result to non-homogeneous measures. We work not just with the Lebesgue measure but with measures μ in $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$, $n \in (0,d]$. The range of exponents in the Cotlar type inequality depend on n. Without assuming buffer we get the full range of exponents $p, q \in (1,2]$ for measures with

* Corresponding author.
E-mail addresses: henri.martikainen@helsinki.fi (H. Martikainen), michail.mourgoglou@ehu.eus (M. Mourgoglou), xtolsa@mat.uab.cat (X. Tolsa).

1 H.M. was supported by the Academy of Finland through the grant 266262: Multiparameter dyadic harmonic analysis and probabilistic methods. He is a member of the Finnish Centre of Excellence in Analysis and Dynamics Research.

2 *Current address:* Departamento de Matemàticas, Universidad del País Vasco, Aptdo. 644, 48080 Bilbao, Spain and Ikerbasque, Basque Foundation for Science, Bilbao, Spain.

3 Research of M.M. was supported by the ERC grant 320501 of the European Research Council (FP7/2007–2013).
4 X.T. was supported by the ERC grant 320501 of the European Research Council (FP7/2007–2013) and partially supported by MTM-2016-77635-P, MDM-2014-044 (MICINN, Spain), and by Marie Curie ITN MAnET (FP7-607647).

https://doi.org/10.1016/j.jfa.2017.12.013

$n \le 1$, and in general we get $p, q \in [2 - \epsilon(n), 2]$, $\epsilon(n) > 0$. Consequences for (non-homogeneous) local Tb theorems are discussed.

1. Introduction

Let μ be a Radon measure on $\mathbb{R}^d$. We say that a function b_Q is an $L^p(\mu)$-admissible test function on a cube $Q \subset \mathbb{R}^d$ (with constant B_1), if

(1) $\operatorname{spt} b_Q \subset Q$,
(2) $\mu(Q) = \int_Q b_Q \, d\mu$, and
(3) $\int_Q |b_Q|^p \, d\mu \le B_1 \mu(Q)$.

A long standing problem (even for the Lebesgue measure $\mu = dx$) asks whether the L^2 boundedness of a Calderón–Zygmund operator T follows if we are given $p, q \in (1, \infty)$, and for every cube Q an $L^p(\mu)$-admissible test function b_Q so that

$$\int_Q |Tb_Q|^{q'} \, d\mu \lesssim \mu(Q)$$

and an $L^q(\mu)$-admissible test function p_Q so that

$$\int_Q |T^* p_Q|^{p'} \, d\mu \lesssim \mu(Q).$$

In the case that both exponents are simultaneously small, i.e. $p, q < 2$ (or even $p < 2 = q$), this is still not known in this original form. However, Hytönen–Nazarov [6] showed in the Lebesgue measure case that the L^2 boundedness follows if one assumes the *buffered* testing conditions

$$\int_{2Q} |Tb_Q|^{q'} \, dx + \int_{2Q} |T^* p_Q|^{p'} \, dx \lesssim |Q|.$$

Notice that the estimate over $2Q$ is in fact equivalent to the same estimate over the whole space $\mathbb{R}^d$. A key thing in the Lebesgue measure case is that if $1/p + 1/q \le 1$ (which includes the case $p = q = 2$), then the original testing conditions automatically imply the stronger buffered testing conditions by Hardy's inequality. The non-homogeneous version for $p = q = 2$ (without buffer) is by the first named author and Lacey [7].

The need for the buffer assumption is related to delicate problems in passing from maximal truncations to the original operator. In the Hytönen–Nazarov paper [6] the