Journal of Functional Analysis 274 (2018) 1284–1305

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

The character and the wave front set correspondence in the stable range

Tomasz Przebinda

University of Oklahoma, Norman, OK, USA

ARTICLE INFO

Article history:
Received 10 February 2017
Accepted 3 January 2018
Available online 5 January 2018
Communicated by P. Delorme

MSC:
primary 22E45
secondary 22E46, 22E30

Keywords:
Howe correspondence
Characters
Wave front set

ABSTRACT

We relate the distribution characters and the wave front sets of unitary representation for real reductive dual pairs of type I in the stable range.

Contents

E-mail addresses: tprzebinda@gmail.com, tprzebinda@ou.edu.

https://doi.org/10.1016/j.jfa.2018.01.002

1. Introduction

In the late seventies Roger Howe formulated his theory of rank for irreducible unitary representations Π of any connected cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$, see [12]. The symplectic group has a maximal parabolic subgroup P with the Levi factor isomorphic to $\mathrm{GL}_n(\mathbb{R})$ and the unipotent radical N isomorphic as a Lie group to the space of the symmetric $n \times n$ matrices with the addition. In particular any connected cover of N splits. The Spectral Theorem implies that the restriction of Π to N is supported on the union of some $\mathrm{GL}_n(\mathbb{R})$-orbits in the dual of N, which may be viewed as the space of the symmetric forms on $\mathbb{R}^n$. The rank of Π is the maximal rank of a symmetric form in this support.

A surprising result is that the representations Π of rank $r < n$ are very special. The support of $\Pi|_{\mathrm{N}}$ is a single $\mathrm{GL}_n(\mathbb{R})$-orbit of forms β of signature (p, q) with $p + q = r$. Furthermore, Π factors through a double cover $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ and remains irreducible when restricted to some other maximal parabolic subgroup $\widetilde{\mathrm{P}}_1 \subseteq \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. The Levi factor of P_1 is isomorphic to $\mathrm{GL}_r(\mathbb{R}) \times \mathrm{Sp}_{2(n-r)}(\mathbb{R})$ and the unipotent radical N_1 is a two-step nilpotent group. The isometry group of a fixed form β is isomorphic to $\mathrm{O}_{p,q} \subseteq \mathrm{GL}_r(\mathbb{R})$. According to [13, Theorem 1.3], there is an irreducible unitary representation Π' of $\widetilde{\mathrm{O}}_{p,q}$ such that $\Pi|_{\widetilde{\mathrm{P}}_1}$ is induced from a representation involving Π' of the subgroup $(\widetilde{\mathrm{O}}_{p,q} \times \widetilde{\mathrm{Sp}}_{2(n-r)}(\mathbb{R}))\mathrm{N}_1 \subseteq \widetilde{\mathrm{P}}_1$. The argument is based on the Stone von Neumann Theorem [30], the theory of the Weil Representation [32] and the Mackey Imprimitivity Theorem, [20].

In particular the operators of $\Pi|_{\widetilde{\mathrm{P}}_1}$ are as explicit as the operators of Π'. However the remaining operators remain obscure. Fortunately there is a different description of the representations Π and Π'.

The groups $(\mathrm{O}_{p,q}, \mathrm{Sp}_{2(n-r)}(\mathbb{R}))$ form a dual pair in $\mathrm{Sp}_{2n}(\mathbb{R})$ and there is Howe's correspondence for all real dual pairs $(\mathrm{G}, \mathrm{G}')$, [14, Theorem 1]. As shown by Jian-Shu Li in his thesis, the representations Π and Π' are in Howe's correspondence. Li extended Howe's theory of rank to all dual pairs of type I and proved that it provides a bijection of representations of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$ equal to Howe's correspondence, see [18] and [17]. The condition of low rank is transformed to the dual pair being in the stable range, with G' – the smaller member. Now the operators $\Pi(g)$, $g \in \widetilde{\mathrm{G}}$, are much better understood because the Weil representation is known explicitly, see [23] or section 2 below, for a coordinate free approach.

Nevertheless an explicit description of all the $\Pi(g)$, $g \in \widetilde{\mathrm{G}}$, seems out of reach. Instead one may try to describe the distribution character Θ_Π of Π, [8], in terms of $\Theta_{\Pi'}$. This approach has a solid foundation, because for the dual pair $(\mathrm{U}_n, \mathrm{U}_n)$ the correspondence of the characters is governed by the Cauchy determinant identity, see [22, Introduction]. In fact [22, Definition 2.17] provides a candidate $\Theta'_{\Pi'}$ for Θ_Π in terms of $\Theta_{\Pi'}$. (For a more precise version see [4, Formula (7)].) Let $\mathrm{G}'_1 \subseteq \mathrm{G}'$ be the Zariski identity component. Here is our first theorem.