Accepted Manuscript

The Brezis-Nirenberg problem for the curl-curl operator

Jarosław Mederski

PII: S0022-1236(18)30003-X
DOI: https://doi.org/10.1016/j.jfa.2017.12.012
Reference: YJFAN 7936

To appear in: *Journal of Functional Analysis*

Received date: 22 April 2017
Accepted date: 22 December 2017

THE BREZIS-NIRENBERG PROBLEM FOR THE CURL-CURL OPERATOR

JAROSŁAW MEDERSKI

ABSTRACT. We look for solutions $E:\Omega\to\mathbb{R}^3$ of the problem

$$\begin{cases}\nabla\times(\nabla\times E)+\lambda E=|E|^{p-2}E & \text{in }\Omega\\ \nu\times E=0 & \text{on }\partial\Omega\end{cases}$$

on a bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$, where $\nabla\times$ denotes the curl operator in $\mathbb{R}^3$. The equation describes the propagation of the time-harmonic electric field $\Re\{E(x)e^{i\omega t}\}$ in a nonlinear isotropic material Ω with $\lambda=-\mu\varepsilon\omega^2\leq 0$, where μ and ε stand for the permeability and the linear part of the permittivity of the material. The nonlinear term $|E|^{p-2}E$ with $p>2$ is responsible for the nonlinear polarisation of Ω and the boundary conditions are those for Ω surrounded by a perfect conductor. The problem has a variational structure and we deal with the critical value p, for instance, in convex domains Ω or in domains with $\mathcal{C}^{1,1}$ boundary, $p=6=2^*$ is the Sobolev critical exponent and we get the quintic nonlinearity in the equation. We show that there exist a cylindrically symmetric ground state solution and a finite number of cylindrically symmetric bound states depending on $\lambda\leq 0$. We develop a new critical point theory which allows to solve the problem, and which enables us to treat more general anisotropic media as well as other variational problems.

MSC 2010: Primary: 35Q60; Secondary: 35J20, 58E05, 35B33, 78A25

Key words: time-harmonic Maxwell equations, perfect conductor, ground state, variational methods, strongly indefinite functional, Nehari-Pankov manifold, Brezis-Nirenberg problem, critical exponent.

1. Introduction

The following equation

$$\nabla\times\left(\mu^{-1}\nabla\times\mathcal{E}\right)+\varepsilon\partial_t^2\mathcal{E}=-\partial_t^2\mathcal{P}_{NL}.$$

describes the propagation of the electric field $\mathcal{E}$ in a nonlinear bounded medium Ω with the permeability μ, the linear part of the permittivity ε and the nonlinear polarisation $\mathcal{P}_{NL}$; see Saleh and Teich [34]. In the time-harmonic case the fields $\mathcal{E}$ and $\mathcal{P}_{NL}$ are of the form $\mathcal{E}(x,t)=$