ARTICLE IN PRESS YJFAN:7840

Journal of Functional Analysis ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Moving-centre monotonicity formulae for minimal submanifolds and related equations

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A R T I C L E I N F O

Article history:
Received 3 May 2017
Accepted 13 July 2017
Available online xxxx
Communicated by S. Brendle

Keywords:
Monotonicity
Minimal submanifolds
Geometric flows

A B S T R A C T

Monotonicity formulae play a crucial role for many geometric PDEs, especially for their regularity theories. For minimal submanifolds in a Euclidean ball, the classical monotonicity formula implies that if such a submanifold passes through the centre of the ball, then its area is at least that of the equatorial disk. Recently Brendle and Hung proved a sharp area bound for minimal submanifolds when the prescribed point is not the centre of the ball, which resolved a conjecture of Alexander, Hoffman and Osserman. Their proof involves asymptotic analysis of an ingeniously chosen vector field, and the divergence theorem.

In this article we prove a sharp 'moving-centre' monotonicity formula for minimal submanifolds, which implies the aforementioned area bound. We also describe similar moving-centre monotonicity formulae for stationary p-harmonic maps, mean curvature flow and the harmonic map heat flow.

0. Introduction

For many geometric partial differential equations, monotonicity formulae play an essential role and their discovery often leads to deep and fundamental results for those

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http://dx.doi.org/10.1016/j.jfa.2017.07.008
0022-1236/

Please cite this article in press as: J.J. Zhu, Moving-centre monotonicity formulae for minimal submanifolds and related equations, J. Funct. Anal. (2017), http://dx.doi.org/10.1016/j.jfa.2017.07.008

systems. Monotonicity is a particularly useful tool in the study of variational problems, and for regularity theory (see for example [3,5,11,14,13,24,27] and references therein). These formulae often control the evolution of energy-type quantities with respect to changes in scale, or time.

An important example is the classical monotonicity formula for minimal submanifolds – critical points of the area functional – which states:

Proposition 0.1. *Let Σ^k be a minimal submanifold in $\mathbb{R}^n$. Then so long as $\partial\Sigma \cap \overline{B^n_r} = \emptyset$, we have*

$$\frac{d}{dr}\left(r^{-k}|\Sigma \cap B^n_r|\right) = r^{-k-1} \int\limits_{\Sigma\cap\partial B^n_r} \frac{|x^\perp|^2}{|x^T|} \geq 0. \tag{0.1}$$

Here $B^n_r = B^n(0,r)$ denotes the Euclidean ball of radius r about the origin in $\mathbb{R}^n$. Thus the area ratio $r^{-k}|\Sigma \cap B^n_r|$ is monotone on balls with fixed centre, and so comparing to the limiting density as $r \searrow 0$ yields that any minimal submanifold $\Sigma^k \subset B^n_r$ with $\partial\Sigma \subset \partial B^n_r$, which passes through the origin, satisfies the sharp area bound

$$\frac{|\Sigma \cap B^n_r|}{r^k} \geq |B^k_1|, \tag{0.2}$$

with equality if and only if Σ is a flat k-disk.

In the case that the minimal submanifold $\Sigma^k \subset B^n_r$ does not necessarily pass through the centre of the ball, Alexander, Hoffman and Osserman [2] conjectured (see also [20]) the following sharp area bound, which has recently been proven in full generality by Brendle and Hung [7] (see also Corollary 1.5). Alexander and Osserman had previously proven the conjecture only in the case of simply connected surfaces [1].

Theorem 0.2 *([7]). Let Σ^k be a minimal submanifold in the ball B^n_r with $\partial\Sigma \subset \partial B^n_r$. Then*

$$\frac{|\Sigma \cap B^n_r|}{(r^2 - d^2)^{\frac{k}{2}}} \geq |B^k_1|, \tag{0.3}$$

where $d = d(0,\Sigma)$ is the distance from Σ to the centre of the ball.

The proof of Theorem 0.2 by Brendle–Hung involves the choice of a clever, but somewhat geometrically mysterious, vector field W. They apply the divergence theorem to W away from small balls $B_\epsilon(y)$, where $y \in \Sigma \cap B_r$, and obtain the estimate in the limit as $\epsilon \to 0$.

In this paper, we show that the area bound (0.3) in fact arises from a sharp 'moving-centre' monotonicity formula, in which the centres of the extrinsic balls are allowed to move, and the scale is adjusted in a particular manner:

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