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Frames arising from irreducible solvable actions I

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A R T I C L E I N F O

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In this work, we provide a unified method for the construction of reproducing systems arising from unitary irreducible representations of some solvable Lie groups. In contrast to other well-known techniques such as the coorbit theory, the generalized coorbit theory and other discretization schemes, we make no assumption on the integrability or square-integrability of the representations of interest. Moreover, our scheme produces explicit constructions of frames with precise frame bounds. As an illustration of the scope of our results, we highlight that a large class of representations which naturally occur in wavelet theory and time–frequency analysis is handled by our scheme. For example, the affine group, the generalized Heisenberg groups, the shearlet groups, solvable extensions of vector groups and various solvable extensions of non-commutative nilpotent Lie groups are a few examples of groups whose irreducible representations are handled by our method. The class of representations studied in this work is described as follows. Let G be a simply connected, connected, completely solvable Lie group with Lie algebra $\mathfrak{g} = \mathfrak{p} + \mathfrak{m}$. Next, let π be an infinite-dimensional unitary irreducible representation of G obtained by inducing a character from a closed normal subgroup $P = \exp \mathfrak{p}$ of G. Additionally, we assume that $G = P \rtimes M$, $M = \exp \mathfrak{m}$ is a closed subgroup of G, $d\mu_M$ is a fixed Haar measure on the solvable Lie group M and there exists a linear functional $\lambda \in \mathfrak{p}^*$ such that the representation $\pi = \pi_\lambda = \mathrm{ind}_P^G(\chi_\lambda)$ is realized as acting in $L^2(M, d\mu_M)$. Making no assumption on the integrability of π_λ, we describe explicitly a discrete subset Γ of G and a vector $\mathbf{f} \in L^2(M, d\mu_M)$ such that $\pi_\lambda(\Gamma)\mathbf{f}$ is a tight frame for $L^2(M, d\mu_M)$. We also construct compactly supported smooth

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functions $\mathbf{s}$ and discrete subsets $\Gamma \subset G$ such that $\pi_\lambda(\Gamma)\mathbf{s}$ is a frame for $L^2(M, d\mu_M)$.

Contents

1. Introduction and preliminaries

Let G be a locally compact group, and let π be a strongly continuous unitary irreducible representation of G acting in an infinite-dimensional Hilbert space $\mathcal{H}_\pi$. Next, let Γ be a discrete subset of G and fix $\mathbf{f} \in \mathcal{H}_\pi$. We say that $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a **frame** for $\mathcal{H}_\pi$ if there exist positive constants $a \leq b$ (frame bounds) such that

$$a\,\|\mathbf{h}\|^2_{\mathcal{H}_\pi} \leq \sum_{\gamma\in\Gamma} \left|\langle \mathbf{h},\pi(\gamma)\mathbf{f}\rangle_{\mathcal{H}_\pi}\right|^2 \leq b\,\|\mathbf{h}\|^2_{\mathcal{H}_\pi}$$

for any vector $\mathbf{h}$ in $\mathcal{H}_\pi$. The frame operator S is defined as

$$S\mathbf{h} = \sum_{\gamma\in\Gamma} \langle \mathbf{h},\pi(\gamma)\mathbf{f}\rangle_{\mathcal{H}_\pi}\,\pi(\gamma)\mathbf{f} \qquad (\mathbf{h}\in\mathcal{H}_\pi)$$

and if $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is a frame for $\mathcal{H}_\pi$ then S is invertible and every vector $\mathbf{h}$ in $\mathcal{H}_\pi$ admits the expansion

$$\mathbf{h} = \sum_{\gamma\in\Gamma} \left\langle \mathbf{h}, S^{-1}\pi(\gamma)\,\mathbf{f}\right\rangle_{\mathcal{H}_\pi} \pi(\gamma)\,\mathbf{f}$$

with convergence in the norm of $\mathcal{H}_\pi$. If $a = b$ then $\{\pi(\gamma)\mathbf{f} : \gamma \in \Gamma\}$ is called a **tight frame** and every vector $\mathbf{h} \in \mathcal{H}_\pi$ admits the simpler series expansion

$$\mathbf{h} = \sum_{\gamma\in\Gamma} \left\langle \mathbf{h},\pi(\gamma)\frac{\mathbf{f}}{\sqrt{a}}\right\rangle_{\mathcal{H}_\pi} \pi(\gamma)\frac{\mathbf{f}}{\sqrt{a}}.$$

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