ARTICLE IN PRESS YJFAN:7916

Journal of Functional Analysis ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Optimal convergence rate and regularity of nonrelativistic limit for the nonlinear pseudo-relativistic equations

Woocheol Choi [a], Younghun Hong [b,*], Jinmyoung Seok [c,*]

[a] *Department of Mathematics Education, Incheon National University, Incheon 22012, Republic of Korea*
[b] *Department of Mathematics, Yonsei University, Seoul 03722, Republic of Korea*
[c] *Department of Mathematics, Kyonggi University, Suwon 16227, Republic of Korea*

A R T I C L E   I N F O

Article history:
Received 3 February 2017
Accepted 10 November 2017
Available online xxxx
Communicated by B. Schlein

MSC:
35J60
35J10

Keywords:
Nonrelativistic limit
Pseudo-relativistic operator
Semilinear elliptic
Ground state

A B S T R A C T

In this paper, we are concerned with the nonrelativistic limit of the following pseudo-relativistic equation with Hartree nonlinearity or power type nonlinearity

$$\left(\sqrt{-\hbar^2c^2\Delta + m^2c^4} - mc^2\right)u + \mu u = \mathcal{N}(u),$$

where c denotes the speed of light. We prove that as $c \to \infty$ the ground states of this equation converges to the ground state of its nonrelativistic counterpart

$$-\frac{\hbar^2}{2m}\Delta u + \mu u = \mathcal{N}(u)$$

with convergence rate $1/c^2$ in every H^s norms. Moreover, we show that this rate is optimal.

* Corresponding authors.
E-mail addresses: choiwc@inu.ac.kr (W. Choi), younghun.hong@yonsei.ac.kr (Y. Hong), jmseok@kgu.ac.kr (J. Seok).

https://doi.org/10.1016/j.jfa.2017.11.006
0022-1236/© 2017 Elsevier Inc. All rights reserved.

1. introduction

We consider the pseudo-relativistic equation

$$i\hbar\partial_t\psi = (\sqrt{-\hbar^2c^2\Delta + m^2c^4} - mc^2)\psi - \mathcal{N}(\psi), \tag{1.1}$$

where

$$\psi = \psi(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$$

is the wave function, $n \geq 1$ is the space dimension, $\hbar$ is the reduced Planck constant, $c > 0$ denotes the speed of light and $m > 0$ represents the particle mass. The operator $\sqrt{-\hbar^2c^2\Delta + m^2c^4}$ is defined as the Fourier multiplier with symbol $\sqrt{\hbar^2c^2|\xi|^2 + m^2c^4}$. The nonlinear term $\mathcal{N}(\psi)$ is assumed to be either of the power-type

$$\mathcal{N}(\psi) = |\psi|^{p-2}\psi$$

or of the Hartree-type

$$\mathcal{N}(\psi) = (|x|^{-1} * |\psi|^2)\psi.$$

The equation (1.1) is referred as the pseudo-relativistic nonlinear Schrödinger equation (NLS) (or nonlinear Hartree equation (NLH), respectively) when the nonlinearity is of the power-type (or the Hartree-type, respectively). Throughout the paper, we always assume that

$$n \geq 1 \text{ and } 2 < p < \tfrac{2n}{n-1}$$

(we set $\frac{2n}{n-1} = \infty$ when $n = 1$) for the power-type nonlinearity, which makes $\mathcal{N}(u)$ be $H^{1/2}(\mathbb{R}^n)$ subcritical, while we assume that $n = 3$ when we refer $\mathcal{N}(u)$ to the Hartree nonlinearity.

This model is considered as a relativistic correction to the nonrelativistic counterpart, because by the Taylor series expansion of the symbol

$$\sqrt{\hbar^2c^2|\xi|^2 + m^2c^4} - mc^2 = mc^2\Big(\sqrt{1 + \frac{\hbar^2|\xi|^2}{m^2c^2}} - 1\Big) = \frac{\hbar^2|\xi|^2}{2m} - \frac{\hbar^4|\xi|^4}{8m^3c^2} + \cdots, \tag{1.2}$$

the nonrelativistic equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi - \mathcal{N}(\psi) \tag{1.3}$$

formally approximates the equation (1.1) in the nonrelativistic regime

$$|\mathbf{p}| = \hbar|\xi| \ll mc.$$

Please cite this article in press as: W. Choi et al., Optimal convergence rate and regularity of nonrelativistic limit for the nonlinear pseudo-relativistic equations, J. Funct. Anal. (2018), https://doi.org/10.1016/j.jfa.2017.11.006